Ordinary Differential Equations
End of Topic Test

Test Structure: **Total: 350 points**

- **Section A:** 8 multiple choice questions. 15 points available.
- **Section B:** 12 short-form questions. 135 points available.
- **Section C:** 9 long-form questions. 200 points available.
- **Cheat sheet:** **1)** Trigonometric identities **2)** Derivatives and integrals **3)** Linear differential equations **4)** Numerical methods **5)** Frobenius method **6)** Stability criteria **7)** Laplace transforms **8)** Z-transforms

Test Topics

- **First-order ODEs:** separable, linear, exact, homogeneous, Bernoulli.
- **Second-order ODEs:** linear (non-)homogeneous, Cauchy-Euler, series solutions (power, Frobenius), Laplace transforms, convolutions.
- **Numerical methods:** Euler's method, Heun's method, Runge-Kutta 4th order
- **Systems of ODEs:** order reduction, matrix methods, phase plane, stability.
- **Difference equations:** linear recurrence relations, Z-transforms.
- **Modelling:** using basic physics (mechanics), linear time invariant systems.

Guidance

For this test, you **should:**

- ✓ have your own plain or lined paper
- ✓ have a scientific calculator
- ✓ use the cheat sheet provided
- ✓ check the worked solutions only when you have tried every question
- ✓ take regular breaks e.g. between Sections A, B and C

For this test, you **should not:**

- ✗ use online calculators, graph plotters or computer code
- ✗ refer to any notes or other cheat sheets
- ✗ give up on a question until you have tried everything you know

There is no time limit, but a good pace is approximately 1 point = 1 minute.

Gradings

Approximately: 50% → pass; 60% → great; 70% → excellent.

Feedback: corrections/questions can be sent to Lorcan at: lorcan.nicholls@cantab.net.

Section A

Multiple Choice Questions

- There are **eight** questions, **all** of which should be answered.
- Choose **one** correct answer from the **four** options: ① ② ③ ④
- A correct answer will receive all points. An incorrect answer will receive zero points.
- A total of **15** points are available in Section A, out of the total of **350** for the test.

A1. The differential equation

$$4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 2^{-x}$$

can be classified as

I	first-order	**II**	second-order
III	linear	**IV**	homogeneous
V	first-degree	**VI**	autonomous

① **I** and **V** only
② **I** only
③ **II** and **III** only
④ **IV**, **V** and **VI** only **[1 point]**

A2. Which statement is true?

① All Bernoulli differential equations are linear and first-order.

② All Cauchy-Euler differential equations are linear and second-order.

③ All autonomous differential equations are homogeneous.

④ All nonlinear differential equations are second-degree or higher.

[1 point]

A3. Evaluate $\int_{-\infty}^{\infty} \left(\delta\left(x + \frac{\pi}{2}\right) + \delta\left(x - \frac{\pi}{2}\right)\right) e^x \sin x \, dx$, where $\delta(\cdot)$ is the Dirac delta function.

① 0

② $2 \cosh \frac{\pi}{2}$

③ $2 \sinh \frac{\pi}{2}$

④ undefined; the integral diverges. **[1 point]**

A4. If $\frac{dy}{dx} = 10 - 4y$ and $y(0) = 0$, then the value of $y(2)$ to two decimal places is

① 0.34

② 2.16

③ 2.38

④ 2.50 **[1 point]**

A5. Find the general solution to the differential equation $\frac{dy}{dx} = \frac{y}{x} - cos^2 \frac{y}{x}$.

① $C\, x^2\, e^{-sin\, x}$

② $x\, ln\left(1 + sin^2 x\right) + C$

③ $x\, tan^{-1}(C - ln\, x)$

④ $\frac{x}{2}\left(1 - x\, sec^2 x\right) + C$ **[2 points]**

A6. If $(x^2 + x)\,dy = \frac{dx}{y}$ then the value of e^{y^2} for all $(x, y) > 0$ is

① proportional to the square of $\frac{x}{1+x}$

② equal to $\left(x + \frac{1}{x}\right)^2$ plus a constant

③ proportional to $\frac{y}{\sqrt{x}}$

④ inversely proportional to xy **[2 points]**

A7. The function $f(t)$ satisfies the equation $\int_0^t f(\tau)\, f(t - \tau)\, d\tau = 16 \sin 4t$ for all $t \geq 0$.

The Laplace transform of $f(t)$ is $F(s)$. Find $F(s)$.

① $\frac{64}{s^2 + 16}$

② $\frac{\pm 8}{s^2 - 16}$

③ $\frac{64}{\sqrt{s^2 - 16}}$

④ $\frac{\pm 8}{\sqrt{s^2 + 16}}$ **[2 points]**

A8. A discrete sequence $\{u\}_n : n \in \mathbf{N}$ satisfies the recurrence relation

$$u_{n+2} = 2u_{n+1} - 2u_n, \qquad u_1 = 1, \quad u_2 = 6.$$

Let $S_n = \sum_{k=1}^{n} u_k$. Find the exact value of S_{100}.

① $5 \times 2^{100} - 1$

② $5 \times \left(2^{50} + 1\right)$

③ $5 \times 2^{50} + 1$

④ $5 \times \left(2^{100} - 1\right)$ **[5 points]**

Section B

Short-Form Questions

- There are **twelve** questions, **all** of which should be answered.
- Show all of your working out clearly.
- You can still obtain partial points for an incorrect final answer if parts of your method were correct.
- A total of **135** points are available in Section B, out of the total of **350** for the test.

B1. Find the general solutions to the following differential equations.

a. $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x, \qquad x \in \mathbb{R}.$$ **[5 points]**

b. $$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = \ln x, \qquad x > 0.$$ **[5 points]**

B2. A real-valued function $y(x)$ satisfies the differential equation **(1)** below.

$$\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}. \qquad \textbf{(1)}$$

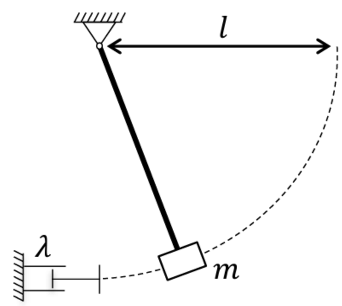
Using the substitution $u = \left(\frac{dy}{dx}\right)^2$, where u is a function of y, show that **(1)** can be transformed into a differential equation for the function $u(y)$, and hence show that the general solution, for arbitrary real constants A and B, is given by

$$y(x) = A\cos^2\left(\sqrt{2}x + B\right).$$

[10 points]

B3. A shock testing machine consists of a mass m on the end of a light rigid rod of length l which swings from a fixed pivot in a vertical plane after it is released from the horizontal position.

The mass makes a head-on collision with a viscous buffer (dashpot) of damping rate λ directly vertically below the pivot as shown. At this instant, the mass is travelling to the left with a speed of $\sqrt{2gl}$.



After first contacting the buffer, the mass displaces the buffer by a small horizontal distance $x << l$ before coming to rest.

Find an approximate expression for x in terms of m, λ, g and l.

State and justify any assumptions made. **[10 points]**

B4. A faulty bit in a computer memory switches its state between the binary values "0" or "1" in any given clock cycle with constant probability p.

Suppose that the state of the bit is measured at cycle $n = 0$, and let y_n be the probability that it is in the same state n clock cycles later.

Find an explicit formula for y_n, valid for all integers $n \geq 0$. **[10 points]**

B5. The variation of y with x satisfies the differential equation below.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 14\frac{\mathrm{d}y}{\mathrm{d}x} + 49y = f(x), \qquad \text{where} \quad f(x) = \begin{cases} 0, & \text{for} \quad x < 0 \\ 1, & \text{for} \quad 0 \leq x \leq 1 \\ 0, & \text{for} \quad x > 1 \end{cases}$$

It is given that y and $\frac{dy}{dx}$ are both zero when $x = 0$.

Find a piecewise expression for $y(x)$ and sketch the graph of y against x, stating the exact maximum value of y and the value of x at which it occurs.

[10 points]

B6. For any $0 < t < 41$, the curve $y = x^3 + 2x^2 - 15x + 5$ intersects the horizontal line $y = t$ three times. Of these three intersections, let the point with the largest x-coordinate be $(f(t), t)$ and the point with the smallest x-coordinate be $(g(t), t)$.

a. Show that the functions $x = f(t)$ and $x = g(t)$ are both distinct solutions to the autonomous differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{3x^2 + 4x - 15}, \qquad 0 < t < 41.$$

[3 points]

b. Let $x(t) = a f(t) + b g(t) + c$, where (a, b, c) are real constants.

Find all (a, b, c) such that $x(t)$ also satisfies the differential equation in part **a)**.

[4 points]

c. If $h(t) = t \times (f(t) - g(t))$, find the value of $\frac{dh}{dt}$ at $t = 5$. **[3 points]**

B7. Consider the differential equation **(1)**,

$$y'' + y = \tan x, \qquad y(0) = y'(0) = 0,$$

a. Use variation of parameters to find the true solution $y(x)$. **[7 points]**

b. The Maclaurin series expansion for $\tan x$ is

$$\tan x \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + ..., \qquad |x| < \frac{\pi}{2}.$$

Consider another differential equation **(2)**,

$$z'' + z = x + \frac{1}{3}x^3 + \frac{2}{15}x^5, \qquad z(0) = z'(0) = 0.$$

Explain whether $z(x)$ is an underapproximation or overapproximation to $y(x)$ on the interval $0 < x < \frac{\pi}{2}$. You do not need to solve for $y(x)$ explicitly.

[3 points]

B8. Consider the coupled system of difference equations

$$\begin{cases} x_{n+1} = \alpha x_n + \frac{1}{2}y_n \\ y_{n+1} = (1-\alpha)x_n + \frac{1}{2}y_n \end{cases}$$

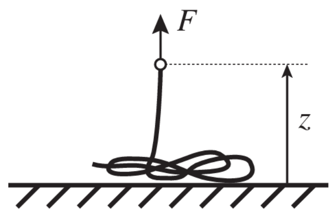
where $0 \leq \alpha \leq 1$ is a constant. The vector $\mathbf{z}_n$ is defined as $\mathbf{z}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$.

a. Given that the eigenvalues of the matrix $\begin{bmatrix} \alpha & \frac{1}{2} \\ 1-\alpha & \frac{1}{2} \end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = \alpha - \frac{1}{2}$, find $\lim_{n \to \infty} \mathbf{z}_n$ if $\mathbf{z}_0 = [1 \quad 0]^T$. **[7 points]**

b. Let $\mathbf{z}_0$ be any unit vector. If $\mathbf{z}_n = [0 \quad 0]^T$ for all $n \geq k$, where k is a finite positive integer, find the value of α.

[3 points]

B9. A heavy chain of length L and mass per unit length ρ is resting coiled on a table.



The chain is being pulled up by one of its ends with a force F which varies with time t. When the chain is in contact with the table, the motion of the chain satisfies the differential equation

$$z\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 + gz = \frac{F(t)}{\rho},$$

where g is the constant acceleration due to gravity.

a. Using the substitution $v = \dfrac{dz}{dt}$, where v is a function of z, show that $v(z)$ satisfies a Bernoulli differential equation, assuming that $v > 0$.

[4 points]

b. Find the minimum value of F_0 such that, if $F(t) = F_0$, then the chain is fully lifted up off the table. What happens to the chain next if this force is maintained?

[6 points]

B10. Consider an asymptotically stable, causal, linear time invariant system with transfer function $H(s)$ in the Laplace s domain, input $x(t)$ and output $y(t)$, where t is time.

a. Which of the following statements is/are true? Explain your answer for each.

I All of the poles s of the Laplace transform of $y(t)$ must satisfy $\mathrm{Re}(s) \leq 0$.

II If $x(t)$ is the Dirac delta function, then the Laplace transform of $y(t)$ is $H(s)$.

III If $\lim_{t \to \infty} x(t) = 0$ then $\lim_{t \to \infty} y(t) = 0$.

IV If $h(t)$ is the inverse Laplace transform of $H(s)$, then $y(t)$ is given by the convolution of $x(t)$ and $h(t)$.

[4 points]

b. Let $\hat{x} : \mathbb{R} \to \mathbb{C}$ and $\hat{y} : \mathbb{R} \to \mathbb{C}$ be complex-valued functions in the time domain.

Assume that the impulse response of the system is real-valued for all time t.

If the LTI system response to $\widehat{x}(t)$ is $\widehat{y}(t)$, prove that the system response to $Re\left[\widehat{x}(t)\right]$ is $Re\left[\widehat{y}(t)\right]$. **[2 points]**

c. Let $x(t) = \cos \omega t$ for all $t \geq 0$, with $x(t) = 0$ otherwise, where ω is a real constant. After a long time $t >> \frac{1}{|\sigma|}$ has elapsed, where σ is the largest real part of any pole s of $H(s)$, the system response $y(t)$ can be written as

$$y(t) \approx A\ x(t - \tau),$$

for some real constants $A \geq 0$ and τ.

By writing $x(t)$ as the real part of a suitable complex-valued exponential function, or otherwise, prove that

$$A = |H(\omega i)| \qquad \text{and} \qquad \tau = -\frac{1}{\omega} \arg H(\omega i).$$

[9 points]

B11. In an epidemic, there are at any particular time x people not yet infected (susceptible) and y people who are ill. The rate at which people become ill is αx. The rates of recovery and death of those who are ill are βy and γy, respectively.

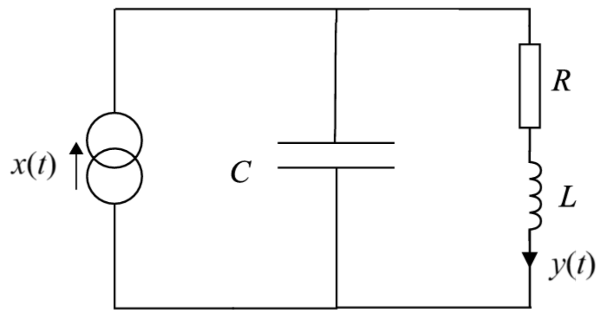
a. If x is initially equal to N and y is initially equal to zero, find an expression for the number of deaths z up to time t from the start of the epidemic.

(Assume that those who recover are immune from further infection.)

[7 points]

b. Find $z(t)$ if $\beta + \gamma = \alpha$ in terms of α and γ. **[8 points]**

B12. The diagram shows an electrical circuit comprising an ideal current source discharging into network of a capacitance C in parallel with a resistance R in series with an inductance L. The values of R, L and C are all positive constants.



At any time $t \geq 0$, the instantaneous current drawn from the source is x and the instantaneous current in the resistor-inductor branch is y. The relationship between x and y can be described as a linear time-invariant system with

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{LC}y = \frac{1}{LC}x, \qquad y(0) = y'(0) = 0.$$

a. Let $Z(s) = \dfrac{Y(s)}{X(s)}$, where $X(s)$ and $Y(s)$ are the Laplace transforms of $x(t)$ and $y(t)$. If $x(t) = x_0 \sin \beta t$ for some positive constants x_0 and β, find $Y(s)$ and $Z(s)$.

[3 points]

b. In testing the circuit, when a DC current input was applied so that $x(t) = u(t)$, where $u(\,\cdot\,)$ is the Heaviside step function, the response $y(t)$ oscillated with decreasing amplitude, eventually stabilising on a positive limiting value as $t \to \infty$.

Find the maximum possible value of R in terms of L and C. **[3 points]**

c. Sketch the pole-zero plot of $Z(s)$, showing the locus of the poles as R varies while L and C remain constant. Describe qualitatively the response of the system to the input $x(t) = x_0 \sin \beta t$ when $R \to 0$ and $s = \beta\mathrm{i}$ is a pole of $Z(s)$.

[4 points]

d. Draw a labelled diagram of a **mechanical system**, consisting of at least one mass m, one linear spring of force constant k and one linear dashpot of damping rate μ, whose dynamics are modelled by the **same differential equation** as the electrical system above.

Identify suitable inputs and outputs $x(t)$ and $y(t)$ for your system, and deduce the required relationships between the variables (m, μ, k) and (R, L, C). **[5 points]**

Section C

Long-Form Questions

- There are **nine** questions, **all** of which should be answered.
- Show all of your working out clearly.
- You can still obtain partial points for an incorrect final answer if parts of your method were correct.
- A total of **200** points are available in Section C, out of the total of **350** for the test.

C1. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + y^9, \qquad 0 < x \leq 1.$$

a. Find the general solution. **[8 points]**

b. Given that $y = 1$ when $x = 1$, find the particular solution. **[3 points]**

c. Using two steps of Heun's (improved Euler's) method with a uniform step size, starting at the point $x = 1$, find an estimate for the value of y when $x = 0.8$.

Give your answer to two significant figures. **[5 points]**

d. Using the solution found in part **b)**, calculate the percentage error in the estimate in part **c)**.

Give your answer to three significant figures. **[2 points]**

e. Give **two** ways you could obtain a better estimate using numerical methods.

[2 points]

C2. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y + \cot x - 1}{\cot x}, \quad y\left(-\frac{\pi}{4}\right) = 0, \quad x \neq n\pi, \;\; n \in \mathbb{Z}.$$

a. Write this differential equation in the form

$$M(x, y)\ dx + N(x, y)\ dy = 0,$$

and state the functions $M(x, y)$ and $N(x, y)$. **[3 points]**

b. Show that this is **not** an exact differential equation. **[3 points]**

c. It is given that

$$I(x)\ M(x, y)\ dx + I(x)\ N(x, y)\ dy = 0$$

is an exact differential equation for some integrating factor function $I : \mathbb{R} \to \mathbb{R}$.

Find all possible functions $I(x)$. **[7 points]**

d. Solve the differential equation to obtain the particular solution $y(x)$.

[7 points]

C3. Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x}\cot x + 2y\csc^2 x = 2\cos x - 2\cos^3 x, \quad y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 0.$$

a. Show that the substitution $y = z\sin x$, where z is a function of x, transforms the above differential equation into

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + z = \sin 2x$$

and give the initial conditions for $z(x)$ in this problem. **[9 points]**

b. For the initial value problem in part **a)**, find the particular solution $z(x)$.

[7 points]

c. Find the particular solution $y(x)$ to the IVP in part **a).**

Give your answer in the form

$$y = a\sin^2 x + b\,(1 - \sin x)\sin 2x$$

where a and b are constants to be found. **[4 points]**

C4. An *autocatalytic reaction* is a chemical system of the form $\mathrm{X} + \mathrm{Y} \rightleftharpoons 2\,\mathrm{Y}$ that can be modelled by the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k_1 xy + k_2 y^2 \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = k_1 xy - k_2 y^2$$

where x is the concentration of the substrate, y is the concentration of the catalyst, t is time, and k_1, k_2 are positive constants.

a. Explain why this system of differential equations **cannot** be written in the form $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = [x \quad y]^{\mathsf{T}}$ and $\mathbf{A}$ is a matrix of constants. **[1 point]**

b. Consider the case of an irreversible autocatalytic reaction, in which $k_2 = 0$.

Show that

$$x(t) = \frac{x_0 + y_0}{1 + \frac{y_0}{x_0} e^{(x_0 + y_0)k_1 t}}$$

where the values of x and y at $t = 0$ are x_0 and y_0 respectively, and obtain a similar simplified expression for $y(t)$. **[7 points]**

c. Sketch the phase plane of the system for the case $k_2 = 0$, identify the nullcline(s) and equilibrium point(s), and describe the system stability.

[4 points]

d. Consider a reversible autocatalytic reaction for which $k_2 \neq 0$.

i) Show that the shape of the field lines of the phase plane does not change from the case $k_2 = 0$, and give a physical justification for this observation.

[2 points]

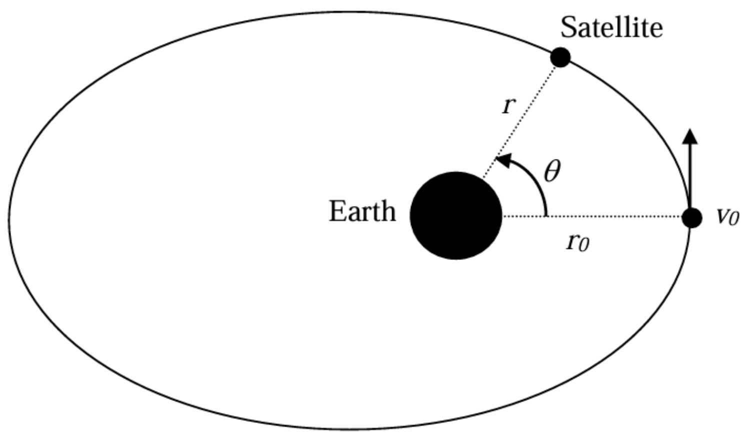
ii) By considering a suitable linearised system, prove that the set of states satisfying $k_2 y = k_1 x$ with $x > 0$ is a line of stable fixed points.

[6 points]

C5. A satellite is in a stable orbit around the Earth. The path of the satellite is elliptical and lies in a plane containing the centre of the Earth. In polar coordinates, with the centre of the Earth as the pole, the distance r and angular position θ of the satellite satisfy the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{gR^2}{r_0^2v_0^2}$$

where g is the surface gravitational acceleration, R is the radius of the Earth, and the initial position and velocity of the satellite are $(r_0, 0)$ and $(0, v_0)$ respectively.



a. Using the substitution $u=\dfrac{1}{r}$, where u and r are functions of θ, obtain the trajectory of the satellite $r(\theta)$. **[7 points]**

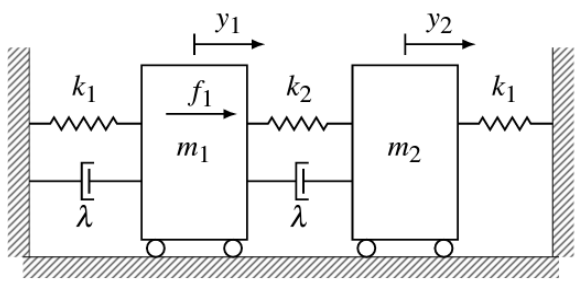
b. Find expressions for the maximum and minimum distances of the satellite from the centre of the Earth throughout its orbit. **[3 points]**

c. Euler's method is to be used to find an approximation for the reciprocal of the radial distance, $u(\theta)$, using a constant step size h. The sequence of approximate values obtained using this iterative scheme is written as $\{u_0, u_1, u_2, \ldots\}$, where u_n is the approximation to $u(nh)$.

i) Find an expression for $U(z)$, the Z-transform of u_n, expressing your answer as a rational function of z, and find expressions for the poles and zeroes of $U(z)$. What happens if $gR^2 = r_0{v_0}^2$? **[9 points]**

ii) Verify the initial value theorem relating u_n and $U(z)$. **[1 point]**

C6. Consider the mass-spring-dashpot system below, consisting of two trolleys of mass m_1 and m_2, joined to each other and two fixed supports by linear viscoelastic elements as shown. The trolleys roll on a frictionless flat surface.



y_1 and y_2 are the displacements of m_1 and m_2 relative to their equilibrium positions, and the springs are unstretched when $y_1 = y_2 = 0$. f_1 is a time-varying force, applied only to m_1, defined as positive in the direction of positive y_1. k_1 and k_2 are force constants for the springs. λ is the damping rate for both dashpots.

a. The system of differential equations for the motion of the trolleys can be written in the form

$$\mathbf{M}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} + \mathbf{K}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}f_1(t)\\ 0\end{bmatrix}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are matrices of constant coefficients relating to the masses, dashpots and springs respectively. ($\dot{y}, \ddot{y}$: first and second time derivatives of y.)

Find the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. **[8 points]**

b. Convert the system of two second-order differential equations into a system of four first-order differential equations. Give your answer in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

where $\mathbf{A}$ is a 4 × 4 matrix of constants and $\mathbf{x}$ is a 4 × 1 state vector. **[6 points]**

c. Assume that the matrix $\mathbf{A}$ has four **complex** eigenvalues of the form $\eta_{1,2} = \alpha \pm \beta\mathrm{i}$ and $\eta_{3,4} = \gamma \pm \delta\mathrm{i}$, with corresponding eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$.

i) If $f_1(t) = 0$ and $\beta \neq \delta \neq 0$, write an expression for the general solution $\mathbf{x}(t)$. **[4 points]**

ii) If $f_1(t) \neq 0$, use variation of parameters to express the particular integral $\mathbf{x}_{\mathrm{PI}}(t)$ in terms of $\mathbf{f}$ and a suitably-defined 4 × 4 matrix $\mathbf{X}$. **[2 points]**

C7.

a. We say that a function $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ is *Lipschitz continuous* if, for all $\mathbf{x}_1 \in \mathbb{R}^n,\ \mathbf{x}_2 \in \mathbb{R}^n$, there exists some $K \in \mathbb{R} : K > 0$ such that

$$|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)| \leq K\,|\mathbf{x}_1 - \mathbf{x}_2|\,.$$

Prove that the function $f(x) = \sqrt{|x|},\ x \in \mathbb{R}$ is **not** Lipschitz continuous.

[4 points]

b. The *Picard-Lindelöf theorem* states that the system of differential equations

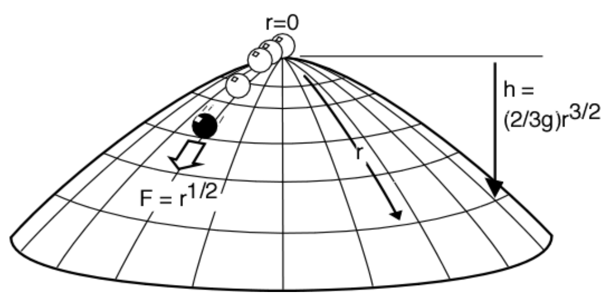
$$\left\{\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x},t),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}'(t_0) = \mathbf{v}_0\right\},\quad \mathbf{x},\mathbf{x}_0,\mathbf{v}_0 \in \mathbb{R}^n;\quad t,t_0 \in \mathbb{R}$$

has a unique solution $\mathbf{x}(t)$ existing for some nonzero interval containing t_0 if and only if the function $\mathbf{f}:\mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous in t and $\mathbf{f}$ is *Lipschitz continuous* in $\mathbf{x}$.

Prove that the IVP $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = \sqrt{|x|},\ x(0) = x'(0) = 0$ has no **unique** solution.

[5 points]

c. *Norton's dome* is a thought experiment in classical physics. The problem concerns a point particle of mass m initially at rest on the top of a smooth surface. The surface is radially symmetric with vertical position $h(r) = \frac{2}{3g}r^{3/2}$ for $0 \leq r < g^2$, where g is the constant acceleration due to gravity.



Applying Newton's second law to this problem yields the equation of motion **(1)**,

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \sqrt{|r|},\qquad r(0) = r'(0) = 0.$$

i) Verify that for any $T \geq 0$, the function $r(t) = \begin{cases} 0 & t \leq T, \\ \frac{1}{144}(t-T)^4 & t > T \end{cases}$

satisfies the differential equation above. **[3 points]**

ii) Philosopher of science John D. Norton interpreted the above solution to mean that, under Newtonian mechanics, it is possible for the particle to remain at the apex ($r = 0$) for an arbitrary amount of time T, and then begin sliding down the dome at $t = T$. He proposes that this means Newtonian mechanics is non-deterministic.

Discuss the validity of this claim. What do you think? **[8 points]**

C8. The nth order *Bessel functions*, $J_n(x)$ and $Y_n(x)$, are defined as two linearly independent basis solutions $y(x)$ to the differential equation

$$x^2\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-n^2)y=0,\quad n\in\mathbb{R}.$$

If $n\in\mathbb{Z}$, then $J_n(x)$ also satisfies $J_n(x)=1$ and $\frac{dJ_n(x)}{dx}=0$ at $x=0$.

a. Show that the power series solution for the **zeroth-order** Bessel function $J_0(x)$ is

$$J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(k!)^2}x^{2k},\qquad x\in\mathbb{R}.$$

[7 points]

b. For any integer $m\geq 1$, the mth *harmonic number* is defined as $H_m=\sum_{j=1}^{m}\frac{1}{j}$.

If the function

$$y_2(x)=J_0(x)\ln x+\sum_{k=1}^{\infty}b_kx^k,\qquad x>0,\quad b_k\in\mathbb{R}$$

also satisfies the Bessel differential equation for $n=0$, use the Frobenius method to find an expression for the coefficients b_k in terms of the harmonic numbers.

[13 points]

c. Briefly explain whether or not the following statements are correct.

i) The functions $y_2(x)$ and $Y_0(x)$ are equal for all $x>0$. **[1 point]**

ii) Fuchs' theorem implies that $y_2(x)$ remains a valid solution to Bessel's differential equation for $n=0$ in the region $x<0$.

[1 point]

d. **i)** Show that $z(x)=x^n J_n(x)$ is a solution to the differential equation

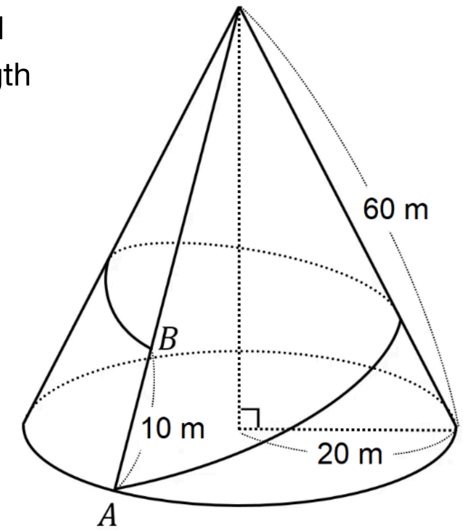
$$x\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+(1-2n)\frac{\mathrm{d}z}{\mathrm{d}x}+xz=0,\quad x>0,\quad n\in\mathbb{R}.$$

[4 points]

ii) Given that $\lim_{x\to 0^+} J_{\frac{1}{2}}(x)$ exists, prove that $J_{\frac{1}{2}}(x)$ is proportional to $\frac{\sin x}{\sqrt{x}}$.

[4 points]

C9. The diagram illustrates a right-circular cone-shaped mountain with base radius 20 metres and slant length 60 metres. The track for a sightseeing train is to be built around the mountain, in which the track starts at a point A at the base of the mountain, and ends at the point B, located 10 metres up the mountain (measured along the slant) above A, as shown.



Define a right-handed spherical coordinate system (r, θ, ϕ) to describe points on the track, centred on the apex. All points on the mountain surface have the same ordinate ϕ and point A has ordinate $\theta = 0$.

a. For a train track whose curve is parameterised in the spherical coordinate system using $r = f(\theta)$ for $0 \leq \theta < 2\pi$, let the total arc length of the track be S.

Find an expression for S in the form $S = \int_0^{2\pi} g(r, u)\ d\theta$, where $u = \dfrac{dr}{d\theta}$.

You may use the fact that in spherical coordinates, the differential element for a position vector $\mathbf{r}$ in terms of the coordinates and orthonormal basis vectors is

$$d\mathbf{r} = dr\ \hat{\mathbf{r}} + r \sin\phi\ d\theta\ \hat{\boldsymbol{\theta}} + r\ d\phi\ \hat{\boldsymbol{\phi}}$$

[4 points]

b. It is now required that, out of all possible train track curves that could be built starting at A, ending at B and going around the mountain, the track with the **shortest total distance** is chosen for construction. The solution $r(\theta)$ to this functional optimisation problem satisfies the *Euler-Lagrange equation*,

$$\frac{\partial g}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial g}{\partial u}\right)$$

where $g(r, u)$ and u are defined in part **a)**.

Solve this differential equation to find the track $r(\theta)$ that satisfies this shortest-distance criterion, and find the value of S for this track. **[20 points]**

c. **Without calculus** (elementary geometry and trigonometry methods only), find the length of the shortest-distance track and use it to verify the value of S found in part **b)**.

Show also that the length of the **downhill portion** of the track as the train moves from A to B on this shortest-distance track is exactly $\frac{400}{\sqrt{91}}$ metres. **[6 points]**

End of Test

Questions in this test were sourced from:

B3, B4, B5, B8, B9, B11, B12, C5, C6	Part IA Engineering Exam, University of Cambridge
C2	Differential equations textbook (Nagle, 2003, 4th ed.)
B2, C3	MadasMaths A-level Further Maths Revision Resources
C8	MadasMaths University-level Maths Revision Resources
B6, C9.c	수학 짝수형 (Korean SAT (*Suneung*) Math Section)

Cheat Sheet

1. TRIGONOMETRIC IDENTITIES

Sine, Cosine and Tangent:

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$

$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$

Hyperbolic Sine, Hyperbolic Cosine and Hyperbolic Tangent:

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}$

$\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}$

$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}$

$\cosh x - \cosh y = -2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$

$2 \sinh x \sinh y = \cosh(x + y) - \cosh(x - y)$

$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y)$

$2 \sinh x \cosh y = \sinh(x + y) + \sinh(x - y)$

2. DERIVATIVES AND INTEGRALS

Function, $f(x)$	**Derivative**, $f'(x)$	**Integral**, $F(x)$ $(+C)$
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$
$\tan x$	$\sec^2 x$	$\ln\|\sec x\|$
$\sec x$	$\sec x \tan x$	$\ln\|\sec x + \tan x\| = \ln\|\tan \frac{x}{2} + \frac{\pi}{4}\|$
$\csc x$	$-\csc x \cot x$	$-\ln\|\csc x + \cot x\| = \ln\|\tan \frac{x}{2}\|$
$\cot x$	$-\csc^2 x$	$\ln\|\sin x\|$
$\sinh x$	$\cosh x$	$\cosh x$
$\cosh x$	$\sinh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$	$\ln \cosh x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$	$2\tan^{-1} \tanh \frac{x}{2} = \tan^{-1} \sinh x$
$\text{csch}\, x$	$-\text{csch}\, x \coth x$	$-\ln\|\text{csch}\, x + \coth x\| = \ln\|\tanh \frac{x}{2}\|$
$\coth x$	$-\text{csch}^2 x$	$\ln\|\sinh x\|$

Function, $f(x)$	**Derivative**, $f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$, $\coth^{-1} x$	$\frac{1}{1-x^2}$

Function, $f(x)$	**Integral**, $F(x)$ $(+C)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$
$\frac{a}{\sqrt{x^4-a^2x^2}}$	$\sec^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1} \frac{x}{a}$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$
$\frac{1}{a^2-x^2}$ $(\|x\| < a)$	$\frac{1}{a} \tanh^{-1} \frac{x}{a}$
$\frac{1}{a^2-x^2}$ $(\|x\| > a)$	$\frac{1}{a} \coth^{-1} \frac{x}{a}$

3. LINEAR DIFFERENTIAL EQUATIONS

Particular Integrals for Nonhomogeneous Differential Equations

$f(x)$	Trial function
1	C
x^n, for integer n	$C x^n + D x^{n-1} + \ldots + C_0$
k^x	$C k^x$
e^{kx}	$C e^{kx}$
$x e^{kx}$	$(Cx + D) e^{kx}$
$x^n e^{kx}$	$(C x^n + D x^{n-1} + \ldots + C_0) e^{kx}$
$\sin px$ or $\cos px$	$C \sin px + D \cos px$
$e^{kx} \sin px$ or $e^{kx} \cos px$	$(C \sin px + D \cos px) e^{kx}$
$x^n e^{kx} \sin px$ or $x^n e^{kx} \cos px$	$(C x^n + D x^{n-1} + \ldots + C_0)(C_S \sin px + C_C \cos px) e^{kx}$

For nonhomogeneous difference equations, replace x with the index n in the above.

Variation of Parameters

For linear nonhomogeneous second-order differential equations, $ay'' + by' + cy = f(x)$:

$$y_{PI}(x) = y_1 \int \frac{y_2 f(x)}{W(x)}\, dx - y_2 \int \frac{y_1 f(x)}{W(x)}\, dx, \quad \text{where} \quad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

For linear nonhomogeneous systems of differential equations, $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$:

$$\mathbf{x}_{PI}(t) = \mathbf{X} \int \mathbf{X}^{-1} \mathbf{f}(t)\, dt, \quad \text{where} \quad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

Complementary Solutions to Linear Systems of Differential Equations

For a 2 × 2 homogeneous autonomous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the general solution is

$$\mathbf{x}(t) = \begin{cases} c_1 e^{\lambda_1 t} \mathbf{u}_1 + c_2 e^{\lambda_2 t} \mathbf{u}_2 & \text{if } \lambda_{1,2} \text{ are real} \\ c_1 e^{\alpha t}(\mathbf{u}_1 \cos \beta t + \mathbf{u}_2 \sin \beta t) + c_2 e^{\alpha t}(\mathbf{u}_1 \cos \beta t - \mathbf{u}_2 \sin \beta t) & \text{if } \lambda_{1,2} = \alpha \pm \beta i \text{ are complex} \\ c_1 e^{\lambda t} \mathbf{u} + c_2 e^{\lambda t}(\mathbf{u} t + \mathbf{v}), \text{ for any } \mathbf{v} : (\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{u} & \text{if } \lambda \text{ is a repeated defective eigenvalue} \end{cases}$$

where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{u}$ is the corresponding eigenvector.

4. NUMERICAL METHODS

For a first-order differential equation of the form $\frac{dy}{dx} = f(x, y)$, iterating with $x_{n+1} = x_n + h$:

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Heun's Method:

$$\hat{y}_{n+1} = y_n + hf(x_n, y_n)$$
$$y_{n+1} = y_n + \frac{1}{2}h\left(f(x_n, y_n) + f(x_{n+1}, \hat{y}_{n+1})\right)$$

Runge-Kutta 4th-order Method:

$$y_{n+1} = y_n + \frac{1}{6}h\left(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}\right)$$
$$k_{n1} = f(x_n, y_n), \qquad k_{n2} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}),$$
$$k_{n3} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}), \qquad k_{n4} = f(x_n + h, y_n + hk_{n3})$$

5. FROBENIUS METHOD

For a differential equation $y'' + p(x)y' + q(x)y = 0$, the indicial equation is

$$k(k-1)+u_0k+v_0 = 0, \text{ where } u_0 = \lim_{x\to x_0}(x-x_0)p(x), \quad v_0 = \lim_{x\to x_0}(x-x_0)^2q(x).$$

Let the solutions to the indicial equation be k_1 and k_2. The general solution has two linearly independent power series solutions y_1 and y_2 given by the Frobenius series:

Case 1: $k_1 - k_2$ **is not an integer**

$$y_1 = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+k_1}, \qquad y_2 = \sum_{n=0}^{\infty} b_n(x - x_0)^{n+k_2}.$$

Case 2: k **is a repeated root**

$$y_1 = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+k}, \qquad y_2 = y_1 \ln|x - x_0| + \sum_{n=1}^{\infty} b_n(x - x_0)^{n+k}.$$

Case 3: $k_1 - k_2$ **is a nonzero integer**

$$y_1 = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+k_1}, \qquad y_2 = ry_1 \ln|x - x_0| + \sum_{n=0}^{\infty} b_n(x - x_0)^{n+k_2}.$$

6. STABILITY CRITERIA FOR LINEAR SYSTEMS

For a continuous-time linear autonomous system modelled by $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$,

Equilibrium Type	**Eigenvalues of $\mathbf{A}$**	**Stability**
Node	Real λ, same signs	$\lambda < 0 \rightarrow$ stable $\lambda > 0 \rightarrow$ unstable
Saddle point	Real λ, opposite signs	depends on initial conditions
Centre / Limit Cycle	λ purely imaginary	marginally stable
Focus / Spiral	Complex λ, $\mathrm{Re}\{\lambda\} \neq 0$	$\mathrm{Re}\{\lambda\} < 0 \rightarrow$ stable $\mathrm{Re}\{\lambda\} > 0 \rightarrow$ unstable
Degenerate Node	Repeated	$\lambda > 0 \rightarrow$ stable
Lines of Equilibria	One eigenvalue $\lambda = 0$	$\lambda < 0 \rightarrow$ stable

For a continuous-time linear system with transfer function $G(s)$,

- The poles of $G(s)$ with $\mathrm{Re}(s) < 0$ are asymptotically stable.
- The simple poles of $G(s)$ with $\mathrm{Re}(s) = 0$ are marginally stable.
- All other poles are unstable.

For a discrete-time linear system with transfer function $G(z)$,

- The poles of $G(z)$ with $|z| < 1$ are asymptotically stable.
- The simple poles of $G(z)$ with $|z| = 1$ are marginally stable.
- All other poles are unstable.

7. LAPLACE TRANSFORMS

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$f(t)$, $t \geq 0$	$F(s)$, $s \in \mathbf{C}$	$f(t)$, $t \geq 0$	$F(s)$, $s \in \mathbf{C}$
$1 = u(t)$	$\frac{1}{s}$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$\delta(t)$	1	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$u(t - a)$	$\frac{e^{-as}}{s}$	$e^{-at}\,x(t)$	$X(s + a)$
$\delta(t - a)$	e^{-as}	$u(t - a)\,x(t - a)$	$e^{-as}\,X(s)$
t	$\frac{1}{s^2}$	$t\,x(t)$	$-\frac{dX}{ds}$
t^n	$\frac{n!}{s^{n+1}}$	$a\,x(t) + b\,y(t)$	$a\,X(s) + b\,Y(s)$
e^{-at}	$\frac{1}{s+a}$	$\frac{dx}{dt}$	$s\,X(s) - x(0)$
$t^n\,e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$\frac{d^2x}{dt^2}$	$s^2\,X(s) - s\,x(0) - x'(0)$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{d^nx}{dt^n}$	$s^n\,X(s) - \sum_{k=0}^{n-1} s^{n-1-k}\,x^{(k)}(0)$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\int_0^t x(\tau)\,d\tau$	$\frac{X(s)}{s}$
$t \sin \omega t$	$\frac{2\omega s}{\left(s^2 + \omega^2\right)^2}$	$\int_0^t x(\tau)\,y(t - \tau)\,d\tau$	$X(s)\,Y(s)$
$t \cos \omega t$	$\frac{s^2 - \omega^2}{\left(s^2 + \omega^2\right)^2}$	$x(t)\sum_{n=0}^{\infty} \delta(t - an)$	$\sum_{n=0}^{\infty} x(an)\,e^{-ans}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$		
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$		

($u(\cdot)$: Heaviside step function, $\delta(\cdot)$: Dirac delta function.)

Initial value theorem: $f(0^+) = \lim_{s \to \infty} s\,F(s)$

Final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$, if $F(s)$ is asymptotically stable.

8. Z-TRANSFORMS

$$F(z) = \mathcal{Z}\{f_n\} = \sum_{k=0}^{\infty} f_k z^{-k}$$

f_n, $n = 0, 1, 2\ldots$	$F(z)$ $z \in \mathbf{C}$	f_n, $n = 0, 1, 2\ldots$	$F(z)$, $z \in \mathbf{C}$
1	$\frac{1}{1 - z^{-1}}$	y_{n+1}	$z\,Y(z) - z\,y_0$
an	$\frac{az^{-1}}{(1 - z^{-1})^2}$	y_{n-1}	$z^{-1}\,Y(z) + y_{-1}$
$\frac{(n+m-1)!}{n!\,(m-1)!} = {}^{n+m-1}C_n$	$(1 - z^{-1})^{-m}$	y_{n+2}	$z^2\,Y(z) - z^2\,y_0 - z\,y_1$
e^{-an}	$\frac{1}{1 - e^{-a}\,z^{-1}}$	y_{n-2}	$z^{-2}\,Y(z) + z^{-1}y_{-1} + y_{-2}$
$\sin \omega n$	$\frac{z^{-1}\sin\omega}{1 - 2z^{-1}\cos\omega + z^{-2}}$	$y_n^{\ *}$	$Y(z^{\ *})^{\ *}$
$\cos \omega n$	$\frac{1 - z^{-1}\cos\omega}{1 - 2z^{-1}\cos\omega + z^{-2}}$	$\sum_{k=0}^{n} y_k$	$\frac{Y(z)}{1 - z^{-1}}$
$\frac{a\sin(\omega(n+1)) - b\sin\omega n}{\sin\omega}\,a^{n-1}$	$\frac{1 - b\,z^{-1}}{1 - 2az^{-1}\cos\omega + a^2 z^{-2}}$		
$(a\cos\omega n + b\sin\omega n)\,r^n$	$\frac{a + rz^{-1}(b\sin\omega - a\cos\omega)}{1 - rz^{-1}\cos\omega + r^2 z^{-2}}$		
y_{-n}	$Y(z^{-1})$		
$a^n\,y_n$	$Y(a^{-1}z)$		
$a\,x_n + b\,y_n$	$a\,X(z) + b\,Y(z)$		
$n\,y_n$	$-\,z\frac{dY}{dz}$		
$\sum_{k=0}^{n} x_k\,y_{n-k}$	$X(z)\,Y(z)$		

Initial value theorem: $f_0 = \lim_{z\to\infty} F(z)$

Final value theorem: $\lim_{n\to\infty} f_n = \lim_{z\to 1} (z - 1)\,F(z)$, if $F(z)$ is asymptotically stable.

Sampling of Continuous-Time Signals: for $t \in \mathbf{R} : t \geq 0$, $T \in \mathbf{R} : T \geq 0$, $n \in \mathbf{Z} : n \geq 0$ and $s \in \mathbf{C}$

If $x_n := x(nT)$ then the Z-transform of x_n is the Laplace transform of $x(t)\sum_{n=0}^{\infty}\delta(t - nT)$ with $z = e^{sT}$.