Differential Equations ~ Cheat Sheet

1. TRIGONOMETRIC IDENTITIES

Sine, Cosine and Tangent:

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

$\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

$\sin x - \sin y = 2 \cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

$\cos x + \cos y = 2 \cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

$\cos x - \cos y = -2 \sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$

Hyperbolic Sine, Hyperbolic Cosine and Hyperbolic Tangent:

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$\cosh x = \dfrac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

$\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1+x}{1-x}$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

$\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

$\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

$\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

$\cosh x - \cosh y = -2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

$2 \sinh x \sinh y = \cosh(x + y) - \cosh(x - y)$

$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y)$

$2 \sinh x \cosh y = \sinh(x + y) + \sinh(x - y)$

2. DERIVATIVES AND INTEGRALS

Function, $f(x)$	**Derivative**, $f'(x)$	**Integral**, $F(x)$ $(+C)$
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$
$\tan x$	$\sec^2 x$	$\ln \lvert\sec x\rvert$
$\sec x$	$\sec x \tan x$	$\ln \lvert\sec x + \tan x\rvert = \ln \lvert\tan \frac{x}{2} + \frac{\pi}{4}\rvert$
$\csc x$	$-\csc x \cot x$	$-\ln\lvert\csc x + \cot x\rvert = \ln \lvert\tan \frac{x}{2}\rvert$
$\cot x$	$-\csc^2 x$	$\ln\lvert\sin x\rvert$
$\sinh x$	$\cosh x$	$\cosh x$
$\cosh x$	$\sinh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$	$\ln \cosh x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$	$2 \tan^{-1} \tanh \frac{x}{2} = \tan^{-1} \sinh x$
$\text{csch}\, x$	$-\text{csch}\, x \coth x$	$-\ln\lvert\text{csch}\, x + \coth x\rvert = \ln \lvert\tanh \frac{x}{2}\rvert$
$\coth x$	$-\text{csch}^2 x$	$\ln \lvert\sinh x\rvert$

Function, $f(x)$	**Derivative**, $f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$, $\coth^{-1} x$	$\frac{1}{1-x^2}$

Function, $f(x)$	**Integral**, $F(x)$ $(+C)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$
$\frac{a}{\sqrt{x^4-a^2x^2}}$	$\sec^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1} \frac{x}{a}$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$
$\frac{1}{a^2-x^2}$ $(\lvert x\rvert < a)$	$\frac{1}{a} \tanh^{-1} \frac{x}{a}$
$\frac{1}{a^2-x^2}$ $(\lvert x\rvert > a)$	$\frac{1}{a} \coth^{-1} \frac{x}{a}$

3. LINEAR DIFFERENTIAL EQUATIONS

Particular Integrals for Nonhomogeneous Differential Equations

$f(x)$	Trial function
1	C
x^n, for integer n	$C x^n + D x^{n-1} + \ldots + C_0$
k^x	$C k^x$
e^{kx}	$C e^{kx}$
$x e^{kx}$	$(Cx + D) e^{kx}$
$x^n e^{kx}$	$(C x^n + D x^{n-1} + \ldots + C_0) e^{kx}$
$\sin px$ or $\cos px$	$C \sin px + D \cos px$
$e^{kx} \sin px$ or $e^{kx} \cos px$	$(C \sin px + D \cos px) e^{kx}$
$x^n e^{kx} \sin px$ or $x^n e^{kx} \cos px$	$(C x^n + D x^{n-1} + \ldots + C_0)(C_S \sin px + C_C \cos px) e^{kx}$

For nonhomogeneous difference equations, replace x with the index n in the above.

Variation of Parameters

For linear nonhomogeneous second-order differential equations, $ay'' + by' + cy = f(x)$:

$$y_{PI}(x) = y_1 \int \frac{y_2 f(x)}{W(x)}\, dx - y_2 \int \frac{y_1 f(x)}{W(x)}\, dx, \quad \text{where} \quad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

For linear nonhomogeneous systems of differential equations, $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$:

$$\mathbf{x}_{PI}(t) = \mathbf{X} \int \mathbf{X}^{-1} \mathbf{f}(t)\, dt, \quad \text{where} \quad \mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

Complementary Solutions to Linear Systems of Differential Equations

For a 2 × 2 homogeneous autonomous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the general solution is

$$\mathbf{x}(t) = \begin{cases} c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2 & \text{if } \lambda_{1,2} \text{ are real} \\ c_1 e^{\alpha t}(\mathbf{u}_1 \cos\beta t + \mathbf{u}_2 \sin\beta t) + c_2 e^{\alpha t}(\mathbf{u}_1 \cos\beta t - \mathbf{u}_2 \sin\beta t) & \text{if } \lambda_{1,2} = \alpha \pm \beta i \text{ are complex} \\ c_1 e^{\lambda t}\mathbf{u} + c_2 e^{\lambda t}(\mathbf{u}t + \mathbf{v}), \text{ for any } \mathbf{v} : (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{u} & \text{if } \lambda \text{ is a repeated defective eigenvalue} \end{cases}$$

where λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{u}$ is the corresponding eigenvector.

4. NUMERICAL METHODS

For a first-order differential equation of the form $\frac{dy}{dx} = f(x, y)$, iterating with $x_{n+1} = x_n + h$:

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Heun's Method:

$$\hat{y}_{n+1} = y_n + hf(x_n, y_n)$$
$$y_{n+1} = y_n + \frac{1}{2}h\left(f(x_n, y_n) + f(x_{n+1}, \hat{y}_{n+1})\right)$$

Runge-Kutta 4th-order Method:

$$y_{n+1} = y_n + \frac{1}{6}h\left(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}\right)$$
$$k_{n1} = f(x_n, y_n), \qquad k_{n2} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}),$$
$$k_{n3} = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}), \qquad k_{n4} = f(x_n + h, y_n + hk_{n3})$$

5. FROBENIUS METHOD

For a differential equation $y'' + p(x)y' + q(x)y = 0$, the indicial equation is

$$k(k-1) + u_0 k + v_0 = 0, \quad \text{where } u_0 = \lim_{x \to x_0}(x - x_0)p(x), \quad v_0 = \lim_{x \to x_0}(x - x_0)^2 q(x).$$

Let the solutions to the indicial equation be k_1 and k_2. The general solution has two linearly independent power series solutions y_1 and y_2 given by the Frobenius series:

Case 1: $k_1 - k_2$ **is not an integer**

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+k_1}, \qquad y_2 = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+k_2}.$$

Case 2: k **is a repeated root**

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+k}, \qquad y_2 = y_1 \ln|x - x_0| + \sum_{n=1}^{\infty} b_n (x - x_0)^{n+k}.$$

Case 3: $k_1 - k_2$ **is a nonzero integer**

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+k_1}, \qquad y_2 = r y_1 \ln|x - x_0| + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+k_2}.$$

6. STABILITY CRITERIA FOR LINEAR SYSTEMS

For a continuous-time linear autonomous system modelled by $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$,

Equilibrium Type	Eigenvalues of $\mathbf{A}$	Stability
Node	Real λ, same signs	$\lambda < 0 \rightarrow$ stable $\lambda > 0 \rightarrow$ unstable
Saddle point	Real λ, opposite signs	depends on initial conditions
Centre / Limit Cycle	λ purely imaginary	marginally stable
Focus / Spiral	Complex λ, $\mathrm{Re}\{\lambda\} \neq 0$	$\mathrm{Re}\{\lambda\} < 0 \rightarrow$ stable $\mathrm{Re}\{\lambda\} > 0 \rightarrow$ unstable
Degenerate Node	Repeated	$\lambda > 0 \rightarrow$ stable
Lines of Equilibria	One eigenvalue $\lambda = 0$	other $\lambda < 0 \rightarrow$ stable

For a continuous-time linear system with transfer function $G(s)$,

- The poles of $G(s)$ with $\mathrm{Re}(s) < 0$ are asymptotically stable.
- The simple poles of $G(s)$ with $\mathrm{Re}(s) = 0$ are marginally stable.
- All other poles are unstable.

For a discrete-time linear system with transfer function $G(z)$,

- The poles of $G(z)$ with $|z| < 1$ are asymptotically stable.
- The simple poles of $G(z)$ with $|z| = 1$ are marginally stable.
- All other poles are unstable.

7. LAPLACE TRANSFORMS

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$f(t)$, $t \geq 0$	$F(s)$, $s \in \mathbf{C}$	$f(t)$, $t \geq 0$	$F(s)$, $s \in \mathbf{C}$
$1 = u(t)$	$\frac{1}{s}$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$\delta(t)$	1	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$u(t - a)$	$\frac{e^{-as}}{s}$	$e^{-at}\, x(t)$	$X(s + a)$
$\delta(t - a)$	e^{-as}	$u(t - a)\, x(t - a)$	$e^{-as}\, X(s)$
t	$\frac{1}{s^2}$	$t\, x(t)$	$-\frac{dX}{ds}$
t^n	$\frac{n!}{s^{n+1}}$	$a\, x(t) + b\, y(t)$	$a\, X(s) + b\, Y(s)$
e^{-at}	$\frac{1}{s+a}$	$\frac{dx}{dt}$	$s\, X(s) - x(0)$
$t^n\, e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$\frac{d^2x}{dt^2}$	$s^2\, X(s) - s\, x(0) - x'(0)$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{d^nx}{dt^n}$	$s^n\, X(s) - \sum_{k=0}^{n-1} s^{n-1-k}\, x^{(k)}(0)$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\int_0^t x(\tau)\, d\tau$	$\frac{X(s)}{s}$
$t \sin \omega t$	$\frac{2\omega s}{\left(s^2 + \omega^2\right)^2}$	$\int_0^t x(\tau)\, y(t - \tau)\, d\tau$	$X(s)\, Y(s)$
$t \cos \omega t$	$\frac{s^2 - \omega^2}{\left(s^2 + \omega^2\right)^2}$	$x(t) \sum_{n=0}^{\infty} \delta(t - an)$	$\sum_{n=0}^{\infty} x(an)\, e^{-ans}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$		
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$		

($u(\cdot)$: Heaviside step function, $\delta(\cdot)$: Dirac delta function.)

Initial value theorem: $f(0^+) = \lim_{s \to \infty} s\, F(s)$

Final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, F(s)$, if $F(s)$ is asymptotically stable.

8. Z-TRANSFORMS

$$F(z) = \mathcal{Z}\{f_n\} = \sum_{k=0}^{\infty} f_k z^{-k}$$

f_n, $n = 0, 1, 2\ldots$	$F(z)$ $z \in \mathbf{C}$	f_n, $n = 0, 1, 2\ldots$	$F(z)$, $z \in \mathbf{C}$
1	$\frac{1}{1-z^{-1}}$	y_{n+1}	$z\,Y(z) - z\,y_0$
an	$\frac{az^{-1}}{(1-z^{-1})^2}$	y_{n-1}	$z^{-1}\,Y(z) + y_{-1}$
$\frac{(n+m-1)!}{n!\,(m-1)!} = {}^{n+m-1}C_n$	$\left(1 - z^{-1}\right)^{-m}$	y_{n+2}	$z^2\,Y(z) - z^2\,y_0 - z\,y_1$
e^{-an}	$\frac{1}{1-e^{-a}z^{-1}}$	y_{n-2}	$z^{-2}\,Y(z) + z^{-1}y_{-1} + y_{-2}$
$\sin \omega n$	$\frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega + z^{-2}}$	$y_n{}^*$	$Y(z^*)^*$
$\cos \omega n$	$\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega + z^{-2}}$	$\sum_{k=0}^{n} y_k$	$\frac{Y(z)}{1-z^{-1}}$
$\frac{a\sin(\omega(n+1)) - b\sin\omega n}{\sin\omega}\,a^{n-1}$	$\frac{1-b\,z^{-1}}{1-2az^{-1}\cos\omega + a^2z^{-2}}$		
$(a\cos\omega n + b\sin\omega n)\,r^n$	$\frac{a + rz^{-1}(b\sin\omega - a\cos\omega)}{1-rz^{-1}\cos\omega + r^2z^{-2}}$		
y_{-n}	$Y(z^{-1})$		
$a^n y_n$	$Y(a^{-1}z)$		
$a\,x_n + b\,y_n$	$a\,X(z) + b\,Y(z)$		
$n\,y_n$	$-\,z\,\frac{dY}{dz}$		
$\sum_{k=0}^{n} x_k\,y_{n-k}$	$X(z)\,Y(z)$		

Initial value theorem: $f_0 = \lim_{z\to\infty} F(z)$

Final value theorem: $\lim_{n\to\infty} f_n = \lim_{z\to 1} (z-1)\,F(z)$, if $F(z)$ is asymptotically stable.

Sampling of Continuous-Time Signals: for $t \in \mathbf{R} : t \geq 0$, $T \in \mathbf{R} : T \geq 0$, $n \in \mathbf{Z} : n \geq 0$ and $s \in \mathbf{C}$

If $x_n := x(nT)$ then the Z-transform of x_n is the Laplace transform of $x(t)\sum_{n=0}^{\infty}\delta(t - nT)$ with $z = e^{sT}$.