Ordinary Differential Equations

Solutions to End of Topic Test

Section A

Multiple Choice Questions

15 points available

Answer Key

Question	A1	A2	A3	A4	A5	A6	A7	A8
Answer	②	②	③	④	③	①	④	②
Points	1	1	1	1	2	2	2	5

Worked Solutions:

A1. **Answer:** ② **I** only **[1 pt]**

Working:

I: First order because the highest order derivative is $\frac{dy}{dx}$.

II: Not second order because there are no $\frac{d^2y}{dx^2}$ terms.

III: Non-linear because the $\frac{dy}{dx}$ term is squared.

IV: Non-homogeneous due to the 2^{-x} term.

V: Second degree due to the $\left(\frac{dy}{dx}\right)^2$ term.

VI: Not autonomous due to the 2^{-x} term.

A2. Answer: ② All Cauchy-Euler differential equations are linear and second-order. **[1 pt]**

Working:

Bernoulli DEs ($y' + P(x)\,y = Q(x)\,y^n$) are first-order and **nonlinear**.
Cauchy-Euler DEs ($x^2 y'' + b\,xy' + c\,y = 0$) are second-order and **linear**.
Autonomous DEs $y' = f(y)$ may contain constants so may be **nonhomogeneous.**
Nonlinear DEs may be **any degree**, and first-degree does not imply linear.

A3. Answer: ③ $2\sinh\frac{\pi}{2}$ **[1 pt]**

Working:

Expand:

$$I = \int_{-\infty}^{\infty} \delta\left(x + \frac{\pi}{2}\right) e^x \sin x\, dx + \int_{-\infty}^{\infty} \delta\left(x - \frac{\pi}{2}\right) e^x \sin x\, dx$$

Apply the sifting theorem, where the Dirac delta functions will 'spike' at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ respectively, both of which are within the interval of integration:

$$I = e^{\pi/2} \sin(\tfrac{\pi}{2}) + e^{-\pi/2} \sin(-\tfrac{\pi}{2}) = e^{\pi/2} - e^{-\pi/2} = 2 \times \frac{e^{\pi/2} - e^{-\pi/2}}{2} = 2\sinh\frac{\pi}{2}.$$

A4. Answer: ④ 2.50 **[1 pt]**

Working:

This is a separable differential equation. Separate and integrate with initial conditions:

$$\frac{dy}{dx} = 10 - 4y \;\Rightarrow\; \int_0^y \frac{1}{10-4y}\, dy = \int_0^x dx$$

Evaluating these integrals gives the particular solution:

$$\Rightarrow\quad -\frac{1}{4} \ln\left(1 - \frac{2}{5}y\right) = x \;\Rightarrow\; y = \frac{5}{2}\left(1 - e^{-4x}\right)$$

Therefore,

$$\Rightarrow\; y(2) = \frac{5}{2}\left(1 - e^{-8}\right) \approx 2.50.$$

A5. **Answer:** ③ $x\,tan^{-1}(C - ln\,x)$ **[2 pts]**

Working:

Observe that $\frac{dy}{dx} = \frac{y}{x} - cos^2\frac{y}{x}$ is a homogeneous DE, since the RHS is purely a function of $\frac{y}{x}$. Therefore, substituting $u(x) = \frac{y}{x}$ will lead to the solution.

From the substitution, we have $y = ux \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$.

Sub into the DE: $x\frac{du}{dx} + u = u - cos^2 u \Rightarrow \frac{du}{dx} = \frac{-cos^2 u}{x}$.

This is a separable DE: $\int sec^2 u\,du = \int \frac{-1}{x}\,dx \Rightarrow tan\,u = C - ln\,x$

Rearrange for u: $u = tan^{-1}(C - ln\,x)$

Unsubstitute for y: $y = x\,tan^{-1}(C - ln\,x)$ for any arbitrary constant C.

(This is also equal to $y = -x\,tan^{-1}(ln\,x - C)$, since $tan^{-1}x$ is an odd function.)

A6. **Answer:** ① proportional to the square of $\frac{x}{1+x}$ **[2 pts]**

Working:

This is a separable DE:

$$\int y\,dy = \int \frac{1}{x(x+1)}\,dx \Rightarrow \frac{1}{2}y^2 = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = ln\,\frac{x}{x+1} + C$$

$$\Rightarrow y^2 = 2\,ln\,\frac{x}{x+1} + 2C \Rightarrow e^{y^2} = e^{2C}\left(\frac{x}{x+1}\right)^2 \Rightarrow e^{y^2} \propto \left(\frac{x}{x+1}\right)^2.$$

A7. **Answer:** ④ $\frac{\pm 8}{\sqrt{s^2+16}}$ **[2 pts]**

Working:

The LHS is a convolution of f with itself: $\int_0^t f(\tau)\,f(t-\tau)\,d\tau = (f * f)(t) = 16\,sin\,4t$

Convolution theorem on the LHS: $F(s)\,F(s) = \frac{64}{s^2+16} \Rightarrow F(s) = \frac{\pm 8}{\sqrt{s^2+16}}$.

A8. **Answer:** ② $5 \times \left(2^{50} + 1\right)$ **[5 pts]**

Working:

The given difference equation is $u_{n+2} - 2u_{n+1} + 2u_n = 0$.

Observe that $u_n = \sum_{k=1}^{n} u_k - \sum_{k=1}^{n-1} u_k = S_n - S_{n-1}$.

Substituting this into the difference equation, we get

$$S_{n+2} - S_{n+1} - 2(S_{n+1} - S_n) + 2(S_n - S_{n-1}) = 0$$
$$S_{n+2} - 3\,S_{n+1} + 4\,S_n - 2\,S_{n-1} = 0$$

By iteration, we can find $u_1 = 1,\ u_2 = 6,\ u_3 = 10 \ \Rightarrow\ S_1 = 1,\ S_2 = 7,\ S_3 = 17$.

The characteristic equation is $\lambda^3 - 3\lambda^2 + 4\lambda - 2 = 0$.

By the rational root/factor theorem, we can observe that $(\lambda - 1)$ is a factor of this cubic polynomial. By synthetic division, we can complete the factorisation as

$$(\lambda - 1)(\lambda^2 - 2\lambda + 2) = 0 \ \Rightarrow\ \lambda = 1,\ \lambda = 1 \pm i = \sqrt{2}\,e^{\pm\frac{\pi}{4}i}.$$

General solution: $S_n = A \times 1^n + (\sqrt{2})^n\left(B \times cos\,\frac{\pi n}{4} + C \times sin\,\frac{\pi n}{4}\right)$

Initial conditions:
$1 = A + B + C$
$7 = A + 2C$
$17 = A - 2B + 2C$

Solving the system gives: $A = 5,\ B = -5,\ C = 1$

Particular solution: $S_n = 5 + 2^{n/2}\left(sin\,\frac{\pi n}{4} - 5\,cos\,\frac{\pi n}{4}\right)$

Let $n = 100$: $sin\,25\pi = 0,\ \ cos\,25\pi = cos\,\pi = -1,$
$S_{100} = 5 + 2^{50} \times 5 = 5\left(2^{50} + 1\right)$.

Alternative methods:

1) Find $u_n = 2^{n/2}\left(3\,sin\,\frac{\pi n}{4} - 2\,cos\,\frac{\pi n}{4}\right)$ and evaluate $\sum_{k=1}^{100} u_k$ on a calculator.

2) Using the Z-transform, $U(z) = \frac{5z - 2z^2}{z^2 - 2z + 2} \ \Rightarrow\ S(z) = \frac{1}{1 - z^{-1}}U(z) \ \Rightarrow$ find S_n.

Section B

Short-Form Questions

135 points available

B1.

a. This is a linear 2nd order nonhomogeneous DE with constant coefficients:

Characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \;\Rightarrow\; \lambda = 3,\; \lambda = -1$ **[1 pt]**

Complementary solution: $y_{CF}(x) = A\,e^{3x} + B\,e^{-x}$ **[1 pt]**

Particular integral: $y_{PI}(x) = Cx + D \;\Rightarrow\; y_{PI}'(x) = C \;\Rightarrow\; y_{PI}''(x) = 0$. **[1 pt]**

$\Rightarrow\; -2C - 3(Cx + D) = x$

$\Rightarrow\; -3Cx + (-2C - 3D) = x$

Equate like terms: $\Rightarrow\; C = -\frac{1}{3},\; D = \frac{2}{9}$. **[1 pt]**

General solution: $y(x) = A\,e^{3x} + B\,e^{-x} - \frac{1}{3}x + \frac{2}{9}$. **[1 pt]**

(Total: 5 points)

b. This is a Cauchy-Euler differential equation. We substitute $x = e^u \;\Rightarrow\; u = ln\,x$.
We may derive the expressions for the new derivatives of y as follows:

First derivative: $\frac{du}{dx} = \frac{1}{x} \;\Rightarrow\; \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{x}\frac{dy}{du} \;\Rightarrow\; \frac{dy}{du} = x\frac{dy}{dx}$.

Second derivative: $\frac{d}{dx}\frac{dy}{du} = \frac{d}{dx}\left(x\frac{dy}{dx}\right) \;\Rightarrow\; \frac{d^2y}{du^2}\frac{du}{dx} = \frac{dy}{dx} + x\frac{d^2y}{dx^2}$

$\Rightarrow\; \frac{1}{x}\frac{d^2y}{du^2} = \frac{1}{x}\frac{dy}{du} + x\frac{d^2y}{dx^2} \;\Rightarrow\; \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)$.

On substitution into our DE, or by recalling the formula for the transformation,

$$\frac{d^2y}{du^2} - 2\frac{dy}{du} - 3y = u. \quad \textbf{[2 pts]}$$

This is the same DE as in part **a)**, so $y(u) = A\,e^{3u} + B\,e^{-u} - \frac{1}{3}u + \frac{2}{9}$. **[2 pts]**

Undo substitution $u = \ln x$: $y(x) = A\,x^3 + \frac{B}{x} - \frac{1}{3}\,ln\,x + \frac{2}{9}$. **[1 pt]**

(Total: 5 points)

B2. Let $u(y) = \left(\frac{dy}{dx}\right)^2$.

Differentiate both sides w.r.t. x: $\frac{du}{dx} = 2\,\frac{dy}{dx}\,\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2}$

Chain rule: $\frac{du}{dy} = \frac{du}{dx}\,\frac{dx}{dy} = \frac{du}{dx} \div \frac{dy}{dx} = 2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2} \div \frac{dy}{dx} = 2\,\frac{d^2y}{dx^2}$

$\Rightarrow\ \frac{d^2y}{dx^2} = \frac{1}{2}\,\frac{du}{dy}$

Substitute into DE: $\frac{1}{2}u = 4y^2 + \frac{1}{2}y\,\frac{du}{dy} \ \Rightarrow\ \frac{du}{dy} = \frac{u}{y} - 8y$ **[3 pts]**

$\frac{du}{dy} - \frac{1}{y}u = -8y.$

This is a linear first-order DE in $u(y)$.

Integrating factor: $I(y) = \exp\int -\frac{1}{y}\,dy = e^{-\ln y} = \frac{1}{y}$

General solution: $I(y)\,u(y) = \int I(y) \times -8y\,dy$

$\Rightarrow\ \frac{u(y)}{y} = \int -8\,dy = -8y + A$

$\Rightarrow\ u(y) = Ay - 8y^2$

Unsubstitute: $\left(\frac{dy}{dx}\right)^2 = Ay - 8y^2 \ \Rightarrow\ \frac{dy}{dx} = \pm\sqrt{Ay - 8y^2}$ **[4 pts]**

This is a separable DE in $y(x)$. $\pm\int \frac{1}{\sqrt{Ay - 8y^2}}\,dy = \int dx = x + B.$

To evaluate the integral on the LHS, complete the square and use trig substitution:

$$\frac{1}{\sqrt{Ay-8y^2}} = \frac{1}{\sqrt{-8\left(y^2 - \frac{A}{8}y\right)}} = \frac{1}{\sqrt{-8\left(\left(y-\frac{A}{16}\right)^2 - \frac{A^2}{256}\right)}} = \frac{1}{\sqrt{\frac{A^2}{32} - 8\left(y-\frac{A}{16}\right)^2}} = \frac{1}{2\sqrt{2}}\,\frac{1}{\sqrt{\left(\frac{A}{16}\right)^2 - \left(y-\frac{A}{16}\right)^2}}$$

Using the result $\int \frac{1}{\sqrt{a^2 - z^2}}\,dz = \sin^{-1}\frac{z}{a}$ with $z = y - \frac{A}{16}$ and $a = \frac{A}{16}$,

$$\pm\frac{1}{2\sqrt{2}}\sin^{-1}\frac{y-\frac{A}{16}}{\frac{A}{16}} = x + B \ \Rightarrow\ y = \frac{A}{16}\left(1 \pm \sin 2\sqrt{2}(x+B)\right)$$

$\Rightarrow\ y(x) = \frac{A}{8}\,\frac{1 \pm \sin 2\sqrt{2}(x+B)}{2} = \frac{A}{8}\,\frac{1 \pm \cos\left(2\sqrt{2}(x+B) - \frac{\pi}{2}\right)}{2}$

$\Rightarrow\ y(x) = \frac{A}{8}\cos^2\left(\sqrt{2}(x+B) - \frac{\pi}{4}\right)$ or $\frac{A}{8}\sin^2\left(\sqrt{2}(x+B) - \frac{\pi}{4}\right)$ (double angle formula)

Since $\sin^2 x = \cos^2(x - \pi/2)$, these two solutions are not linearly independent, so the constants A and B can be chosen to make them the same. Therefore, we can take either one solution as the general solution.

Redefine constants: $A \leftarrow \frac{A}{8}$, $B \leftarrow \sqrt{2}B - \frac{\pi}{4}$: $y(x) = A\cos^2\left(\sqrt{2}x + B\right)$. **[3 pts]**

B3. Let $x(t)$ be the displacement of the mass into the buffer (positive to the left) at time t.

While in contact with the buffer, the dashpot exerts a viscous force $F = -\lambda \frac{dx}{dt}$.

Since $x << l$, we can assume the motion is almost horizontal (no vertical component of motion) and we can also neglect the weight force **[1 pt].**

By Newton's second law in the horizontal direction, the equation of motion is

$$F = ma \Rightarrow -\lambda \frac{dx}{dt} = m\frac{d^2x}{dt^2} \Rightarrow mx'' + \lambda x' = 0.$$ **[2 pts]**

Characteristic equation: $m\alpha^2 + \lambda\alpha = 0 \Rightarrow \alpha\left(\alpha + \frac{\lambda}{m}\right) = 0 \Rightarrow \alpha = 0,\ \alpha = -\frac{\lambda}{m}$

General solution: $x(t) = A + B\,e^{-\frac{\lambda}{m}t}$ **[2 pts]**

Initial conditions: $x(0) = 0 \Rightarrow 0 = A + B$

$x'(0) = \sqrt{2gl} \Rightarrow \sqrt{2gl} = \frac{-\lambda}{m}B$

$\Rightarrow A = \frac{m}{\lambda}\sqrt{2gl},\ B = -\frac{m}{\lambda}\sqrt{2gl}$

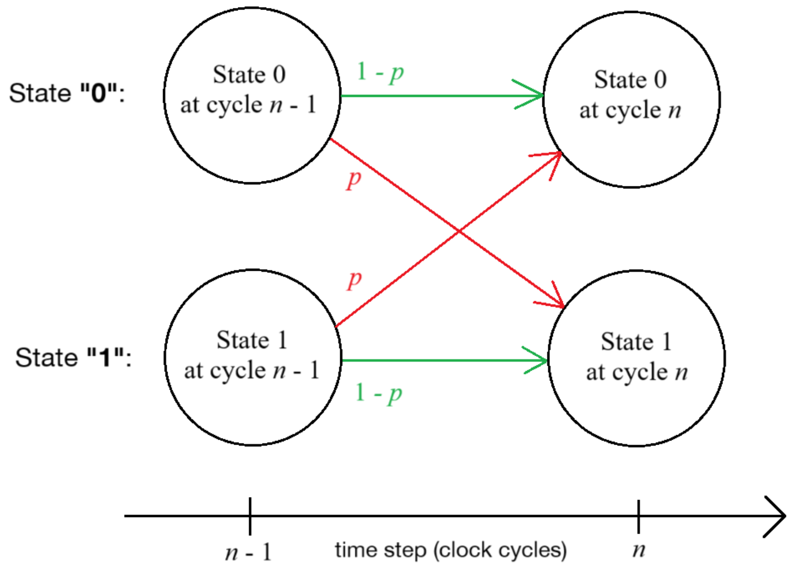
Particular solution: $x(t) = \frac{m}{\lambda}\sqrt{2gl}\left(1 - e^{-\frac{\lambda}{m}t}\right)$ **[3 pts]**

$x(t)$ approaches a horizontal asymptote at $\lim_{t\to\infty} x(t) = \frac{m}{\lambda}\sqrt{2gl}$.

Maximum displacement: $x \approx \frac{m\sqrt{2gl}}{\lambda}$ **[2 pts]**

(Total: 10 points)

B4. Draw the state transition diagram, showing the probabilities of changing states:



For the bit to be in a given state (suppose, state **"0"** without loss of generality as this is a symmetric problem) at cycle n, the bit **must have**:

1. **Either** been in the same state (state "0") at cycle n - 1 and then **stayed the same,**
2. **Or** been in the other state (state "1") at cycle n - 1 and then **flipped**.

Using the 'AND' rule of probability for independent events:

P(Case 1) = P(same state at n - 1 AND did not flip)
= P(same state at n - 1) × P(did not flip)
= P(same state at n - 1) × (1 - P(flipped)) (complementary events)
= $y_{n-1} \times (1 - p)$

P(Case 2) = P(opposite state at n - 1 AND flipped)
= P(opposite state at n - 1) × P(flipped)
= (1 - P(same state at n - 1)) × P(flipped) (complementary events)
= $(1 - y_{n-1}) \times p$

Using the 'OR' rule of probability for disjoint events:

P(same state at n) = P(Case 1 OR Case 2)
= P(Case 1) + P(Case 2)

$\Rightarrow\ y_n = y_{n-1}(1 - p) + (1 - y_{n-1})p$ **[4 pts]**

$\Rightarrow\ y_n + (2p - 1)y_{n-1} = p$

This is a linear nonhomogeneous first-order difference equation.

B4. (continued)

Characteristic equation: $\lambda + (2p - 1) = 0 \Rightarrow \lambda = 1 - 2p$

Complementary solution: $y_n^{(CF)} = A\,(1 - 2p)^n$

Particular integral: $y_n^{(PI)} = B \Rightarrow B + (2p - 1)B = p \Rightarrow B = \frac{1}{2}$

General solution: $y_n = A\,(1 - 2p)^n + \frac{1}{2}$

Initial condition: $y_0 = 1$ (since the initial state is the same by definition)

$\Rightarrow 1 = A + \frac{1}{2} \Rightarrow A = \frac{1}{2}$

Particular solution: $y_n = \frac{1}{2}\left((1 - 2p)^n + 1\right)$, for all integers $n \geq 0$. **[6 pts]**

(Total: 10 points)

B5. IVP: $y'' + 14y' + 49y = f(x),\ \ y(0) = y'(0) = 0.$

The input is $f(x) = u(x) - u(x - 1)$, where $u(t)$ is the Heaviside step function.

To find the step response, let $f(x) = u(x) = 1$ (for $x > 0$, otherwise 0)

Characteristic equation: $\lambda^2 + 14\lambda + 49 = 0 \Rightarrow \lambda = -7$ (repeated) **[1 pt]**

Complementary solution: $y_{CF}(x) = e^{-7x}(A + Bx)$

Particular integral: $y_{PI}(x) = C \Rightarrow 49C = 1 \Rightarrow C = \frac{1}{49}$

General solution: $y_{step}(x) = e^{-7x}(A + Bx) + \frac{1}{49}$ **[1 pt]**

Initial conditions: $y(0) = 0 \Rightarrow 0 = A + \frac{1}{49} \Rightarrow A = \frac{-1}{49}$

$y'(0) = 0 \Rightarrow 0 = -7A + B \Rightarrow B = \frac{-1}{7}$

Step response: $y_{step}(x) = -\frac{1}{49}e^{-7x} - \frac{1}{7}x\,e^{-7x} + \frac{1}{49} = \frac{1}{49}(1 - (1 + 7x)e^{-7x})$
if $x \geq 0$, else 0. **[1 pt]**

Method 1: Linear Superposition

Using the principles of linear superposition, the solutions $y(x)$ with $f(x) = u(x)$ can be found, shifted by 1 unit in x to find the solution with $u(x - 1)$, then subtracted.

Forced input: $f(x) = u(x) - u(x - 1)$

Forced response: $y(x) = y_{step}(x) - y_{step}(x - 1)$

If $x < 0$, then $y_{step}(x)$ and $y_{step}(x)$ are both zero, so $\{y(x) = 0;\ if\ x < 0\}$.

If $0 \leq x < 1$, then $y_{step}(x) = \frac{1}{49}(1 - (1 + 7x)e^{-7x})$ but $y_{step}(x - 1) = 0$, so

$$\left\{y(x) = \frac{1}{49}(1 - (1 + 7x)e^{-7x});\ if\ 0 \leq x < 1\right\}.$$ **[2 pts]**

If $x \geq 1$, then $y_{step}(x) = \frac{1}{49}(1 - (1 + 7x)e^{-7x})$ and $y_{step}(x - 1) = \frac{1}{49}\left(1 - (7x - 6)\,e^{7}\,e^{-7x}\right)$

$$\left\{y(x) = \frac{1}{49}\,e^{-7x}\left((7x - 6)\,e^{7} - (1 + 7x)\right);\ if\ x \geq 1\right\}$$ **[3 pts]**

Method 2: Convolution integral

Impulse response: $g(x) = \frac{dy_{step}}{dx} = x\, e^{-7x}$ **[1 pt]**

Forcing function: $f(x) = u(x) - u(x-1)$

Forced response: $y(x) = (f * g)(x) = \int_0^x f(\tau)\, g(x-\tau)\, d\tau$

If $0 \le x \le 1 \Rightarrow y(x) = \int_0^x (x-\tau)\, e^{-7(x-\tau)}\, d\tau = xe^{-7x}\int_0^x e^{7\tau}\, d\tau - e^{-7x}\int_0^x \tau e^{7\tau}\, d\tau$

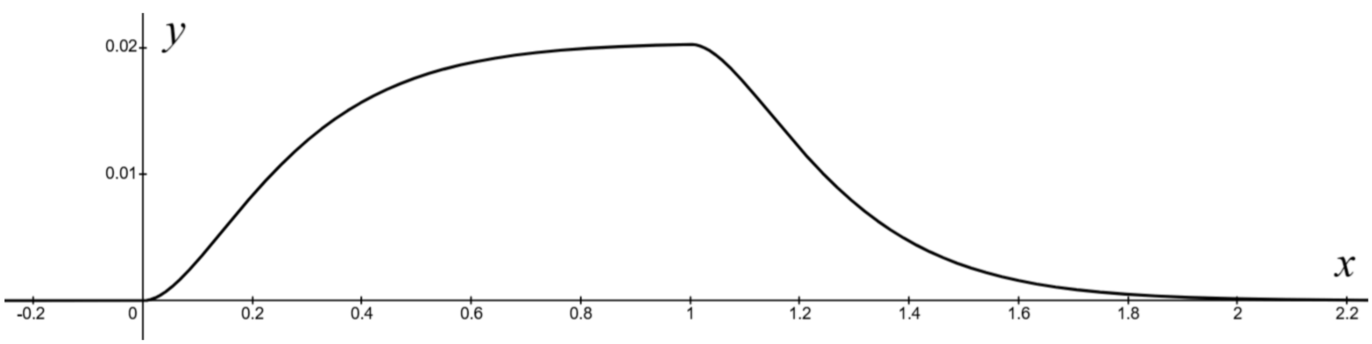
$= \frac{x}{7} - \frac{x}{7}e^{-7x} + \frac{1}{49} - \frac{x}{7} - \frac{1}{49}e^{-7x} = \frac{-x}{7}e^{-7x} + \frac{1}{49} - \frac{1}{49}e^{-7x}$. **[2 pts]**

If $x > 1 \Rightarrow y(x) = \int_0^1 (x-\tau)\, e^{-7(x-\tau)}\, d\tau = \frac{1}{49}e^{-7x}\left(-1 - 7x + e^{7}(7x-6)\right)$.) **[2 pts]**

Therefore, $y(x)$ is a piecewise function with

$$y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{49}\left(1 - e^{-7x}(1+7x)\right), & 0 \le x \le 1 \\ \frac{1}{49}e^{-7x}\left(-1 - 7x + e^{7}(7x-6)\right), & x > 1 \end{cases}$$

Sketch of $y(x)$: **[1 pt]**



The maximum value of y occurs at $x = 1$, at which $y(1) = \frac{1 - 8\,e^{-7}}{49}$.

[1 pt]

(Total: 10 points)

B6.

a. The intersections of the curve with the horizontal line $y = t$ are given by

$$x^3 + 2x^2 - 15x + 5 = t \quad \textbf{[1 pt]}$$

The solutions x satisfy $x_1 = f(t)$ and $x_3 = g(t)$, so the above is the **implicit solution** to a differential equation satisfied by these functions.

Implicitly differentiating both sides with respect to t:

$$\Rightarrow \; 3x^2 \frac{dx}{dt} + 4x \frac{dx}{dt} - 15 \frac{dx}{dt} = 1 \quad \textbf{[1 pt]}$$
$$\Rightarrow \; \frac{dx}{dt}\left(3x^2 + 4x - 15\right) = 1$$
$$\Rightarrow \; \frac{dx}{dt} = \frac{1}{3x^2 + 4x - 15} \quad \textbf{[1 pt]}$$

This differential equation has solutions $x = f(t)$ and $x = g(t)$. **(Total: 3 points)**

b. The differential equation is nonlinear, so it does not satisfy the superposition principle. Therefore, in general, $x(t) = a\,f(t) + b\,g(t)$ is not a solution.
We know that $x = f(t)$ and $x = g(t)$ are solutions, so $(a, b, c) = (1, 0, 0)$ or $(0, 1, 0)$. **[2 pts]**

However, we know that the curve intersects the line three times, and two of them are given by $f(t)$ and $g(t)$, therefore there must exist one further solution.
We need to find out whether we can express this solution in the form $a\,f(t) + b\,g(t)$.

By Vieta's formula, the sum of the cubic polynomial roots is: $f(t) + g(t) + x(t) = -2$

$$\Rightarrow \; x(t) = -\left(f(t) + g(t) + 2\right) = -f(t) - g(t) - 2$$

This cannot be written in the form $a\,f(t) + b\,g(t)$, so there are no other solutions with $c = 0$.
However, when we allow $c \neq 0$, we get **one more solution** $(a, b, c) = (-1, -1, -2)$. **[2 pts]**

(Total: 4 points)

c. Find the intersection points for the case $t = 5$:

$$x^3 + 2x^2 - 15x = 0 \;\Rightarrow\; x(x - 3)(x + 5) = 0 \;\Rightarrow x = 0, 3, -5$$
$$\Rightarrow f(5) = 3, \; g(5) = -5$$

Using the DE, $f'(5) = \frac{1}{3\,f(5)^2 + 4\,f(5) - 15} = \frac{1}{24}$ and $g'(5) = \frac{1}{3\,g(5)^2 + 4\,g(5) - 15} = \frac{1}{40}$.
By product rule, $h'(t) = t(f'(t) - g'(t)) + f(t) - g(t)$, so
$h'(5) = 8 + 5 \times \frac{1}{60} = \frac{97}{12}$. **(Total: 3 points)**

B7.

a. Characteristic equation: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Complementary solution: $y_{CF}(x) = A\cos x + B\sin x$ **[1 pt]**

The linearly independent basis solutions are $y_1 = \cos x$ and $y_2 = \sin x$.

The Wronskian determinant is $W(x) = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$. **[1 pt]**
By variation of parameters, the particular integral is

$$y_{PI}(x) = -\cos x \int \sin x \tan x \, dx + \sin x \int \cos x \tan x \, dx$$

$$= -\cos x \int (\sec x - \cos x)\, dx + \sin x \int \sin x \, dx$$

$$= -\cos x \left(\ln|\sec x + \tan x| - \sin x\right) - \sin x \cos x$$
$$= -\cos x \ln|\sec x + \tan x|.$$ **[3 pts]**

General solution: $y(x) = A\cos x + B\sin x - \cos x \ln|\sec x + \tan x|$.

First derivative of particular integral:
$\frac{d}{dx} \cos x \ln|\sec x + \tan x| = 1 - \sin x \ln|\sec x + \tan x|$

Initial conditions: $y(0) = 0 \Rightarrow 0 = A$
$y'(0) = 0 \Rightarrow 0 = B - 1 \Rightarrow B = 1$

Particular solution: $y(x) = \sin x - \cos x \ln|\sec x + \tan x|$. **[2 pts]**

(Total: 7 points)

b. Observe that the series expansion of $\tan x$ has all positive terms. Therefore, the series approximation to $\tan x$ always **underestimates** the true value of $\tan x$ **[1 pt].**

Therefore, the nonhomogeneous part of the DE has $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 < \tan x$.

We can interpret the DEs as an undamped simple harmonic oscillator subjected to these applied external forces. The period of the motion is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1}} = 2\pi$, so in the interval $0 < x < \frac{\pi}{2}$, we are looking at the first quarter-period of the motion, where the unforced solution is **monotonically increasing [1 pt].**

Therefore, the smaller force value gives a smaller displacement. So, $z(x)$ is an **underapproximation** to $y(x)$ on the interval $0 < x < \frac{\pi}{2}$ **[1 pt].**

(Total: 3 points)

B8.

a. Let $\mathbf{A} = \begin{bmatrix} \alpha & \frac{1}{2} \\ 1-\alpha & \frac{1}{2} \end{bmatrix}$. Observe that the system is described by $\mathbf{z}_{n+1} = \mathbf{A}\mathbf{z}_n$. **[1 pt]**

We are given that the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = \alpha - \frac{1}{2}$.

Eigenvector for $\lambda_1 = 1$: $(\alpha - 1)x + \frac{1}{2}y = 0$, let $x = 1 \rightarrow y = 2(1 - \alpha) \rightarrow \mathbf{v}_1 = [1, 2(1 - \alpha)]^{\mathrm{T}}$.

Eigenvector for $\lambda_2 = \alpha - \frac{1}{2}$: $\frac{1}{2}x + \frac{1}{2}y = 0$, let $x = 1 \rightarrow y = -1 \rightarrow \mathbf{v}_2 = [1, -1]^{\mathrm{T}}$. **[2 pts]**

By iteration, we have $\mathbf{z}_n = \mathbf{A}^n \mathbf{z}_0$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span $\mathbf{R}^2$, we can always find a unique β, γ such that $\mathbf{z}_0 = \beta\mathbf{v}_1 + \gamma\mathbf{v}_2$. Substituting this in, we get

$$\mathbf{z}_n = \mathbf{A}^n\mathbf{z}_0 = \beta\mathbf{A}^n\mathbf{v}_1 + \gamma\mathbf{A}^n\mathbf{v}_2 = \beta\,\lambda_1{}^n\mathbf{v}_1 + \gamma\,\lambda_2{}^n\mathbf{v}_2$$ (by definition of eigenvector: $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$).

In the limit as $n \rightarrow \infty$, since $|\lambda_2| < 1$, we have $\gamma\,\lambda_2{}^n\mathbf{v}_2 \rightarrow \mathbf{0}$, and since $\lambda_1 = 1$, $\mathbf{z}_n \rightarrow \beta\,\mathbf{v}_1$. **[2 pts]**

We are given that $\mathbf{z}_0 = [1, 0]^{\mathrm{T}}$, so equating components in $\mathbf{z}_0 = \beta\mathbf{v}_1 + \gamma\mathbf{v}_2$ gives

$$\{\beta + \gamma = 1,\ \ 2(1 - \alpha)\beta - \gamma = 0\}.$$

Add the equations together to eliminate γ: $(3 - 2\alpha)\beta = 1 \Rightarrow \beta = \frac{1}{3-2\alpha}$.

Therefore, $\lim_{n\rightarrow\infty} \mathbf{z}_n = \frac{1}{3-2\alpha}[1, 2(1 - \alpha)]^{\mathrm{T}}$.

In component form, $\lim_{n\rightarrow\infty} x_n = \frac{1}{3-2\alpha}$ and $\lim_{n\rightarrow\infty} y_n = \frac{2-2\alpha}{3-2\alpha}$. **[2 pts]**

(Total: 7 points)

b. Let $\mathbf{M} = \mathbf{A}^k$. We have $\mathbf{z}_k = \mathbf{M}\mathbf{z}_0 = \mathbf{0}$ for some finite k.

The eigenvalues of $\mathbf{M}$ are $\lambda_1{}^k, \lambda_2{}^k$, with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

For $\mathbf{M}$ to map $\mathbf{z}_0$ to the origin, $\mathbf{z}_0$ must be parallel to the eigenvector $\mathbf{v}_2$, and the corresponding eigenvalue λ_2 must be zero. **[2 pts]**

Therefore, $\lambda_2 = \alpha - \frac{1}{2} = 0 \Rightarrow \alpha = \frac{1}{2}$. **[1 pt]**

Alternative method: If $\mathbf{M}\mathbf{z}_0 = \mathbf{0}$ then $\mathbf{M}^{-1}$ is a singular matrix, so the determinant of $\mathbf{A}$ must be zero. This condition leads to $\alpha = \frac{1}{2}$.

(Total: 3 points)

B9.

a. Substitute $v(z) = \frac{dz}{dt} \Rightarrow \frac{dv}{dz} = \frac{d}{dz}\left(\frac{dz}{dt}\right)$. **[1 pt]**

Since $dz = v\,dt$, this is equal to $\frac{dv}{dz} = \frac{d}{dz}\left(\frac{dz}{dt}\right) = \frac{1}{v}\frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{1}{v}\frac{d^2z}{dt^2}$. **[1 pt]**

Since $v = \frac{dz}{dt} > 0$, $z(t)$ is a monotonically increasing function, so $z(t)$ is invertible. Therefore, there exists a single-valued function $F(z)$ for the force at elevation z.

Applying the substitutions to the DE, we get

$$zv\frac{dv}{dz} + v^2 + gz = \frac{F(z)}{\rho} \Rightarrow \frac{dv}{dz} + \frac{1}{z}v = \left(\frac{F(z)}{\rho z} - g\right)v^{-1} \quad \textbf{[2 pts]}$$

This is a Bernoulli DE for $v(z)$ with $n = -1$.

(Total: 4 points)

b. Let $F(z) = F_0$. The Bernoulli DE to be solved is $\frac{dv}{dz} + \frac{1}{z}v = \left(\frac{F_0}{\rho z} - g\right)v^{-1}$.

Let $u = v^2$. The DE becomes $\frac{du}{dz} + \frac{2}{z}u = 2\left(\frac{F_0}{\rho z} - g\right)$ **[1 pt]**. This is a linear DE.

The integrating factor is $I(z) = z^2$, so the solution is given by $z^2 u = 2\int\left(\frac{F_0}{\rho}z - gz^2\right)dz$.

Therefore, $z^2 u = \frac{F_0}{\rho}z^2 - \frac{2}{3}gz^3 + C \Rightarrow u = \frac{F_0}{\rho} - \frac{2}{3}gz + \frac{C}{z^2}$. **[1 pt]**

Undoing the substitution, $v = \sqrt{u} = \sqrt{\frac{F_0}{\rho} - \frac{2}{3}gz + \frac{C}{z^2}}$. **[1 pt]**

Since $v = \frac{dz}{dt}$ must be finite for all t, including at $t = 0$ when $z = 0$, we must have $C = 0$.

To lift the chain fully, we must have $v \geq 0$ when $z = L$, so $\frac{F_0}{\rho} \geq \frac{2}{3}gL \Rightarrow F_0 \geq \frac{2}{3}\rho gL$. **[1 pt]**

Since the weight of the chain is ρgL, so:

- If $\frac{2}{3}\rho gL \leq F_0 < \rho gL$, then the chain slightly falls back down after lifting. **[1 pt]**
- If $F_0 \geq \rho gL$, then the chain continues being lifted upwards away from the table. **[1 pt]**

(Total: 6 points)

B10.

a. **I**: **False**. Asymptotic stability requires all the poles of $H(s)$ (the transfer function; Laplace transform of the impulse response) to have negative real part. There is no condition on the poles of $Y(s)$. **[1 pt]**

II: **True**. If $x(t) = \delta(t)$ then $X(s) = 1$. By the definition of the transfer function, $Y(s) = H(s)\,X(s) = H(s)$. **[1 pt]**

III: **True**. By the final value theorem, $\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,H(s)\,X(s)$.

Since $H(s)$ is asymptotically stable, $H(s)$ **cannot** have a pole at $s = 0$, so $H(0)$ is a finite value. Therefore,

$\lim_{t\to\infty} y(t) = H(0) \lim_{s\to 0} s\,X(s) = H(0) \lim_{t\to\infty} x(t) = H(0) \lim_{t\to\infty} x(t) = 0$. **[1 pt]**

IV: **True**. By the convolution theorem, $L\{(x * h)(t)\} = X(s)\,H(s)$.
Since $Y(s) = X(s)\,H(s)$, taking ILTs, we have $y(t) = (x * h)(t)$. **[1 pt]**

b. Let $\hat{x}(t) = a(t) + i\,b(t)$, where a and b are real-valued functions.

By linearity, the system response is $\hat{y}(t) = y_a(t) + i\,y_b(t)$, where y_a and y_b are the system responses to inputs $a(t)$ and $b(t)$ respectively. **[1 pt]**

Since $a(t)$ and $b(t)$ are real, and the impulse response $h(t)$ is also real, the responses y_a and y_b are also real, so $Re\left[\hat{y}(t)\right] = y_a(t)$.

Since $a(t) = Re\left[\hat{x}(t)\right]$, the system response to $Re\left[\hat{x}(t)\right]$ is $Re\left[\hat{y}(t)\right]$. **[1 pt]**

(Total: 2 points)

c. Let $\hat{x}(t) = e^{i\omega t}$. By Euler's formula, observe that $x(t) = \cos\omega t = Re[\hat{x}(t)]$. **[1 pt]**
The Laplace transform is $\hat{X}(s) = \frac{1}{s - i\omega}$, so the system response is given by

$$\hat{Y}(s) = H(s)\,\hat{X}(s) = \frac{H(s)}{s - i\omega}.$$ **[1 pt]**

Since this is an LTI system, $H(s)$ must be a rational function of s, with numerator order less than the denominator order. Consider a partial fraction decomposition of $\hat{Y}(s)$:

$$\hat{Y}(s) = \frac{H(s)}{s - i\omega} = \sum_{r=1}^{N} \frac{p_r(s)}{(s - s_r)^{m_r}} + \frac{C}{s - i\omega} \quad \text{where all } \mathrm{Re}(s_r) < 0$$ **[1 pt]**

($p_r(s)$: polynomial functions in s of order up to $m_r - 1$, s_r: rth pole of $H(s)$,
m_r: multiplicity of the pole s_r, N: number of poles in $H(s)$.).

Using the cover-up method, we can let $s = i\omega$ to get $C = H(i\omega)$. **[1 pt]**

The inverse Laplace transform of $\frac{a(s)}{(s - s_r)^m}$ contains terms of the form $t^{m-1}e^{s_r t}$, and since $\lim_{t\to\infty} t^{m-1}e^{s_r t} = 0$ for all $\mathrm{Re}(s_r) < 0$, the inverse Laplace transform of the function $\sum_{r=1}^{N} \frac{a_r(s)}{(s - s_r)^{m_r}}$ tends to zero as $t \to \infty$, with the 'slowest' exponential having time constant $\frac{1}{-\sigma}$ (σ: least negative real part of poles s_r).

Therefore, when $t \to \infty$ (with $t >> \frac{1}{|\sigma|}$ the function $\hat{Y}(s) \approx \frac{C}{s - i\omega} = \frac{H(i\omega)}{s - i\omega}$. **[1 pt]**
Taking the inverse Laplace transform, $\hat{y}(t) \approx H(i\omega)\,e^{i\omega t}$. **[1 pt]**

From the result in part **b)**, the asymptotic solution is $y(t) = Re[\hat{y}(t)]$, so
$y(t) \approx Re[H(i\omega)\,e^{i\omega t}]$. **[1 pt]**

Writing $H(i\omega)$ in modulus-argument form, $H(i\omega) = |H(i\omega)|\,e^{i\arg H(i\omega)}$, so

$y(t) \approx |H(i\omega)|\,Re[e^{i(\omega t + \arg H(i\omega))}]$

$\Rightarrow\ y(t) \approx |H(i\omega)|\cos(\omega t + \arg H(i\omega))$ **[1 pt]**

$\Rightarrow\ y(t) \approx |H(i\omega)|\cos\left(\omega\left(t - \left[-\frac{1}{\omega}\arg H(i\omega)\right]\right)\right) = A\,x(t - \tau)$

where $A = |H(i\omega)|$ and $\tau = -\frac{1}{\omega}\arg H(i\omega)$. **[1 pt]** **(Total: 9 points)**

B11.

a. The differential equations are $\frac{dx}{dt} = -\alpha x$ and $\frac{dy}{dt} = \alpha x - (\beta + \gamma)y$.

The solution to the first DE is $\frac{dx}{dt} = -\alpha x \Rightarrow x(t) = x_0 e^{-\alpha t}$.

Using the initial condition, $x_0 = N \Rightarrow x(t) = N e^{-\alpha t}$.

Subbing in the first DE and this solution for x,

$$\frac{dy}{dt} + (\beta + \gamma)y = \alpha N e^{-\alpha t}$$

This is a linear DE. The integrating factor is $e^{\int (\beta+\gamma)\, dt} = e^{(\beta+\gamma)t}$ and the solution is

$$e^{(\beta+\gamma)t} y = \int \alpha N e^{(\beta+\gamma-\alpha)t} \Rightarrow y = e^{-(\beta+\gamma)t}\left(\frac{\alpha N}{\beta+\gamma-\alpha} e^{(\beta+\gamma-\alpha)t} + C\right)$$

Initial condition: $y(0) = 0 \rightarrow C = \frac{-\alpha N}{\beta+\gamma-\alpha}$

$$\Rightarrow y = \frac{\alpha N}{\beta+\gamma-\alpha} e^{-(\beta+\gamma)t}\left(e^{(\beta+\gamma-\alpha)t} - 1\right) = \frac{\alpha N}{\beta+\gamma-\alpha}\left(e^{-\alpha t} - e^{-(\beta+\gamma)t}\right).$$

To find the number of deaths z, use $\frac{dz}{dt} = \gamma y$, so we integrate y:

$$z = \frac{\alpha\gamma N}{\beta+\gamma-\alpha}\left(\frac{1}{\beta+\gamma}e^{-(\beta+\gamma)t} - \frac{1}{\alpha}e^{-\alpha t}\right) + C$$

The initial condition is $z(0) = 0$, so $C = 0$. Therefore,

$$z(t) = \frac{\alpha\gamma N}{\beta+\gamma-\alpha}\left(\frac{1}{\beta+\gamma}e^{-(\beta+\gamma)t} - \frac{1}{\alpha}e^{-\alpha t}\right).$$

(Total: 7 points)

B11. (continued)

b. Return to the differential equation for y, let $\beta + \gamma = \alpha$:

$$\frac{dy}{dt} + \alpha y = \alpha N e^{-\alpha t}, \quad y(0) = 0$$

This is a linear differential equation. We could use the integrating factor method, but since the RHS $Q(x)$ is an exponential, we can use the easier method of undetermined coefficients.

Characteristic equation: $\lambda + \alpha = 0 \Rightarrow \lambda = -\alpha$

Complementary solution: $y_{CF} = A e^{-\alpha t}$

Particular integral: $y_{PI} = B t e^{-\alpha t}$ (since $e^{-\alpha t}$ is not linearly independent)

$\Rightarrow y_{PI}' = B e^{-\alpha t} - \alpha B t e^{-\alpha t}$

Subbing into the DE, $B e^{-\alpha t} = \alpha N e^{-\alpha t} \Rightarrow B = \alpha N$

General solution: $y = A e^{-\alpha t} + \alpha N t e^{-\alpha t}$

Initial condition: $y(0) = 0 \Rightarrow A = 0$

Particular solution: $y(t) = \alpha N t e^{-\alpha t}$

By integration, $z(t) = \gamma \int y(t)\, dt = \alpha \gamma N \int t e^{-\alpha t}\, dt$

Using integration by parts, $\int t e^{-\alpha t}\, dt = -\frac{t}{\alpha} e^{-\alpha t} - \frac{1}{\alpha^2} e^{-\alpha t} + C$

So, $z(t) = \frac{-\gamma N}{\alpha}\left((1 + \alpha t) e^{-\alpha t} + C'\right)$

Initial conditions: $z(0) = 0 \Rightarrow C' = \frac{\gamma N}{\alpha}$

Therefore, $z(t) = \frac{\gamma N}{\alpha}\left(1 - e^{-\alpha t}(1 + \alpha t)\right)$.

(Total: 8 points)

B12.

a. Laplace transform of both sides: $\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)Y(s) = \frac{1}{LC}X(s)$

From the table of Laplace transforms, $X(s) = x_0 L\{\sin \beta t\} = \frac{x_0 \beta}{s^2+\beta^2}$. **[1 pt]**

Therefore, $Y(s) = \frac{1}{LC}\frac{x_0\beta}{(s^2+\beta^2)\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}$, **[1 pt]**

$Z(s) = \frac{Y(s)}{X(s)} = \frac{1}{LC}\frac{1}{s^2+\frac{R}{L}s+\frac{1}{LC}}$. **[1 pt]**

(Total: 3 points)

b. We are given that the step response contains stable oscillations with a finite steady-state value. Since the impulse response is the time derivative of this, it must also contain stable oscillations, so we know that this is an underdamped system. Therefore, the poles of $Z(s)$ must be complex values.

For complex poles, the discriminant of the characteristic polynomial must be negative:

$$\frac{R^2}{L^2} - \frac{4}{LC} < 0 \Rightarrow \frac{1}{LC} > \frac{R^2}{4L^2}.$$ **[2 pts]**

Therefore, the value of R is bounded from above by $R < 2\sqrt{\frac{L}{C}}$. **[1 pt]**

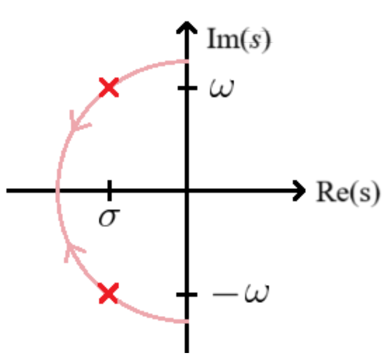
(Total: 3 points)

c. To find the poles of $Z(s)$, we need to set its denominator to zero.

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \Rightarrow s = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$ **[1 pt]**

As R varies, the real and imaginary parts of the poles $s = \sigma \pm \omega i$ are

$$\sigma(R) = Re(s) = \frac{-R}{2L} \text{ and } \omega(R) = Im(s) = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$



Observe that $\sigma^2 + \omega^2 = \frac{1}{LC}$, so the **locus** of the poles is a **circle** in the complex plane with radius $\frac{1}{\sqrt{LC}}$, in the left half plane since $\sigma < 0$.

Therefore, the pole-zero plot is as shown on the left. **[1 pt]**
As R increases from 0 to $2\sqrt{L/C}$, the poles move further left. **[1 pt]**

As $R \to 0$, if $\beta = \omega = \frac{1}{\sqrt{LC}}$ then **resonance** occurs, and the system response $y(t)$ contains **unstable oscillations** that increase in amplitude (diverge) with time. **[1 pt]** **(Total: 4 points)**

B12. (continued)

d. In general, we know that any forced parallel mass-spring-dashpot linear system can be modelled by the nonhomogeneous 2nd order DE:

$$my'' + \mu y' + ky = F(t)$$

where y is the displacement of the mass from its equilibrium position and F is an input force applied to the mass.

Considering our electrical system DE, $y'' + \frac{R}{L}\, y' + \frac{1}{LC}\, y = \frac{1}{LC}\, x$, we get

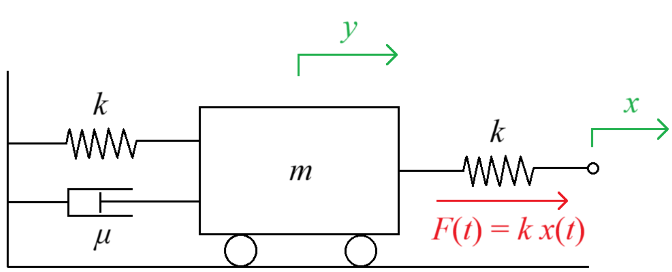
$$my'' + \mu y' + ky = F(t) \quad \Leftrightarrow \quad L\, y'' + R\, y' + \frac{1}{C}\, y = \frac{1}{C}\, x$$

These DEs are the **same** if all of the following conditions are met: **[3 pts]**

- the **mass** m is proportional to the **inductance** L,
- the **damping rate** μ is proportional to the **resistance** R,
- the **spring constant** k is proportional to the **reciprocal of the capacitance** $\frac{1}{C}$,
- the **input force** $F(t)$ is proportional to the **input current divided by capacitance** $\frac{x(t)}{C}$,
- the **constant of proportionality** must be the **same** in all four above cases.

Observe that in the electrical system, the coefficients of the input x and output y are the same, and x and y both represent the same type of quantity (current). We can mirror this in the mechanical system by using both x and y to represent lengths and adding another spring of the same spring constant.

One possible realisation of a mechanical system with the same dynamics is therefore: (assumes no collisions with walls, perfectly smooth horizontal ground, no air resistance)



$$my'' + \mu y' + ky = kx \quad \Leftrightarrow \quad L\, y'' + R\, y' + \frac{1}{C}\, y = \frac{1}{C}\, x$$

where x is the **extension of the spring** on the right-hand side. **[2 pts]**

This must be an **underdamped** system, so we also require $\mu < 2\sqrt{mk}$ for oscillations.

Other systems are also possible - be creative! **(Total: 5 points)**

Section C

Long-Form Questions

250 points available

C1.

a. This is a Bernoulli differential equation.

Write in standard form: $\frac{dy}{dx} - \frac{1}{x}y = y^9$

Let $u = y^{1-9} = y^{-8} \Rightarrow \frac{du}{dx} = -8y^{-9}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{8}y^9\frac{du}{dx}$

$y = u^{-1/8} \Rightarrow y^9 = u^{-9/8} \Rightarrow \frac{dy}{dx} = \frac{-1}{8}u^{-9/8}\frac{du}{dx}$

Substitute into the DE: $\frac{-1}{8}u^{-9/8}\frac{du}{dx} - \frac{1}{x}u^{-1/8} = u^{-9/8}$

Multiply both sides by -$8u^{9/8}$: $\frac{du}{dx} + \frac{8}{x}u = -8$ **[5 pts]**

This is a linear DE. Use the integrating factor $I(x) = e^{\int \frac{8}{x}dx} = e^{8\ln x} = x^8$.

General solution: $x^8 u = \int -8x^8\,dx = \frac{-8}{9}x^9 + A \Rightarrow u = \frac{-8x}{9} + \frac{A}{x^8}$.

[2 pts]

Unsubstitute: $y = \left(\frac{-8x}{9} + \frac{A}{x^8}\right)^{-1/8}$ **[1 pt]**

(Total: 8 points)

b. Initial condition: $1 = \left(-\frac{8}{9} + A\right)^{-1/8} \Rightarrow A = \frac{17}{9}$. **[2 pt]**

Particular solution: $y = \left(\frac{-8x}{9} - \frac{17}{9x^8}\right)^{-1/8} = \left(\frac{9x^8}{17 - 8x^9}\right)^{1/8} = \frac{\sqrt[4]{3}\,x}{\sqrt[8]{17 - 8x^9}}$. **[1 pt]**

(Total: 3 points)

C1. (continued)

c. Use a step size $h = -0.1$ (moving to the left), starting at $x_0 = 1, y_0 = 1$. **[1 pt]**

The gradient satisfies $f(x, y) = \frac{y}{x} + y^9$.

Step 1: $\hat{y}_1 = 1 + h\,f(1, 1) = 0.8$ **[1 pt]**

$y_1 = 1 + \frac{h}{2}(f(1, 1) + f(0.9, 0.8)) = 0.8488446692...$ **[1 pt]**

Step 2: $\hat{y}_2 = 0.8488446692 + h\,f(0.9, 0.8488446692) = 0.7316486996...$ **[1 pt]**

$y_2 = 0.8488446692 + \frac{h}{2}(f(0.9, 0.8488446692) + f(0.8, 0.7316486996))$
$= 0.7415146855...$

$\Rightarrow$ $y(0.8) \approx 0.741515$ (6 sf) **[1 pt]**

(Total: 5 points)

d. Exact value: $y(0.8) = \frac{\sqrt[4]{3} \times 0.8}{\sqrt[8]{17 - 8 \times 0.8^9}} = 0.744913906...$ **[1 pt]**

Percentage error: $\frac{0.744913906 - 0.741515}{0.744913906} = 0.004584158$ $\Rightarrow$ 0.46% error. **[1 pt]**

(Total: 2 points)

e. Any two of the following:

- Use a larger number of steps *or* smaller step size.
- Use a more accurate numerical method e.g. Runge-Kutta 4th order (RK4).
- Use a higher floating point precision in the computations (keep more sig figs).

(Total: 2 points)

C2.

a. Cross-multiply: $\cot x\, dy = (y + \cot x - 1)\, dx$ **[1 pt]**

Subtract: $(1 - y - \cot x)\, dx + (\cot x)\, dy = 0$ **[1 pt]**

Therefore $M(x, y) = 1 - y - \cot x$ and $N(x, y) = \cot x$. **[1 pt]**

(Alternative answer: the negatives of these.)

(Total: 3 points)

b. Condition for exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check each partial derivative: $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(1 - y - \cot x) = -1$ **[1 pt]**

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\cot x) = -\csc^2 x$ **[1 pt]**

Since the partial derivatives are different, the DE is **not** exact. **[1 pt]**

(Total: 3 points)

c. Condition for exactness: $\frac{\partial}{\partial y}(\, I(x)\,(1 - y - \cot x)\,) = -I(x)$ **[1 pt]**

$\frac{\partial}{\partial x}(\, I(x) \cot x\,) = -\csc^2 x\, I(x) + I'(x) \cot x$ **[1 pt]**

These must be equal, so $-\csc^2 x\, I(x) + I'(x)\cot x = -I(x)$

$\Rightarrow I'(x)\cot x = I(x)(\csc^2 x - 1) = I(x)\cot^2 x$

$\Rightarrow I'(x) = I(x)\cot x$ **[2 pts]**

This is a separable DE: $\int \frac{1}{I}\, dI = \int \cot x\, dx \Rightarrow \ln I = \ln \sin x + C$ **[2 pts]**

$I(x) = C \sin x$ **[1 pt]** (redefine $C \leftarrow e^C$)
for any real constant C.

(Total: 7 points)

C2. (continued)

d. Exact DE: $M(x,\ y)\ =\ (1\ -\ y\ -\ cot\ x)\ sin\ x$ and $N(x,\ y)\ =\ cos\ x$. **[1 pt]**

Potential function: $F(x,\ y)\ =\ \int M(x,\ y)\ dx\ =\ (y\ -\ 1)\ cos\ x\ -\ sin\ x\ +\ f(y)$ **[1 pt]**

$F(x,\ y)\ =\ \int N(x,\ y)\ dy\ =\ y\ cos\ x\ +\ f(x)$ **[1 pt]**

Equate: $f(x)\ =\ -\ cos\ x\ -\ sin\ x$ and $f(y)\ =\ 0$ **[1 pt]**

General solution: $F(x,\ y)\ =\ y\ cos\ x\ -\ cos\ x\ -\ sin\ x\ =\ C$ **[1 pt]**

$\Rightarrow\ y\ = \frac{C + sin\ x + cos\ x}{cos\ x} =\ C\ sec\ x\ +\ tan\ x\ +\ 1$ **[1 pt]**

Particular solution: $y\left(-\ \frac{\pi}{4}\right) = 0\ \Rightarrow\ 0 = \sqrt{2}C\ -\ 1\ +\ 1\ \Rightarrow\ C\ =\ 0$

$\Rightarrow\ y\ =\ 1\ +\ tan\ x$. **[1 pt]**

(Total: 7 points)

C3.

a. Let $y = z\sin x \Rightarrow \frac{dy}{dx} = \frac{dz}{dx}\sin x + z\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}\sin x + 2\frac{dz}{dx}\cos x - z\sin x$

Substitute into the given DE:

$$\frac{d^2z}{dx^2}\sin x + 2\frac{dz}{dx}\cos x - z\sin x - 2\left(\frac{dz}{dx}\sin x + z\cos x\right)\cot x + 2z\sin x\csc^2 x$$
$$= 2\cos x - 2\cos^3 x$$ **[3 pts]**

Simplify $\sin x\csc^2 x = \csc x$, $\sin x\cot x = \cos x$ and $\cos x\cot x = \cos^2 x\csc x$:

$$\frac{d^2z}{dx^2}\sin x - z\sin x - 2z\cos^2 x\csc x + 2z\csc x = 2\cos x - 2\cos^3 x$$

Factorise: $\frac{d^2z}{dx^2}\sin x + z\left(2\csc x - 2\cos^2 x\csc x - \sin x\right) = 2\cos x\sin^2 x$

Divide by $\sin x$: $\frac{d^2z}{dx^2} + z\left(2\csc^2 x - 2\cot^2 x - 1\right) = 2\cos x\sin x$

Trig identities: $\frac{d^2z}{dx^2} + z(2 - 1) = \sin 2x$

$\frac{d^2z}{dx^2} + z = \sin 2x$. **[4 pts]**

Initial conditions: $z = y\csc x \Rightarrow z(\frac{\pi}{2}) = y(\frac{\pi}{2})\csc\frac{\pi}{2} = 1 \times 1 = 1$ **[1 pt]**

$\frac{dz}{dx} = \csc x(\frac{dy}{dx} - z\cos x)$

$\Rightarrow z'(\frac{\pi}{2}) = \csc\frac{\pi}{2}\,(y'(\frac{\pi}{2}) - z(\frac{\pi}{2})\cos\frac{\pi}{2}) = 0.$ **[1 pt]**

$\Rightarrow z\left(\frac{\pi}{2}\right) = 1$ and $z'\left(\frac{\pi}{2}\right) = 0.$

(Total: 9 points)

C3. (continued)

b. To solve $z'' + z = sin\, 2x$:

Complementary solution: $z_{CF}(x) = A\, cos\, x + B\, sin\, x$ **[2 pts]**

Particular integral: $z_{PI}(x) = C\, cos\, 2x + D\, sin\, 2x$

$\Rightarrow \quad -4C\, cos\, 2x - 4D\, sin\, 2x + C\, cos\, 2x + D\, sin\, 2x = sin\, 2x$

$\Rightarrow \quad C = 0, \quad -3D = 1 \;\Rightarrow\; D = \frac{-1}{3}$

General solution: $z(x) = A\, cos\, x + B\, sin\, x - \frac{1}{3}\, sin\, 2x$ **[3 pts]**

Initial conditions: $z\left(\frac{\pi}{2}\right) = 1 \;\Rightarrow\; B = 1$ **[1 pt]**

$z'\left(\frac{\pi}{2}\right) = 0 \;\Rightarrow\; -A + \frac{2}{3} = 0 \;\Rightarrow\; A = \frac{2}{3}$ **[1 pt]**

Particular solution: $z(x) = \frac{2}{3}\, cos\, x + sin\, x - \frac{1}{3}\, sin\, 2x$

(Total: 7 points)

c. Unsubstitute: $y = z\, sin\, x \;\Rightarrow\; y = \left(\frac{2}{3}\, cos\, x + sin\, x - \frac{1}{3}\, sin\, 2x\right) sin\, x$ **[1 pt]**

$y = \frac{2}{3}\, sin\, x\, cos\, x + sin^2 x - \frac{1}{3}\, sin\, 2x\, sin\, x$ **[1 pt]**

$y = \frac{1}{3}\, sin\, 2x + sin^2 x - \frac{1}{3}\, sin\, 2x\, sin\, x$

$y = sin^2 x + \frac{1}{3}(1 - sin\, x)\, sin\, 2x.$ **[2 pts]**

(Total: 4 points)

C4.

a. The system contains the product terms xy and y^2, which are nonlinear in the dependent variables. Therefore, they cannot be expressed using matrix multiplication with the given state vector $\mathbf{x} = [x \quad y]^T$. **[1 pt]**

(Total: 1 point)

b. The system is $\frac{dx}{dt} = -k_1 xy,\ \frac{dy}{dt} = k_1 xy \ \Rightarrow\ \frac{dx}{dt} = -\frac{dy}{dt} \ \Rightarrow\ x + y = A$

(or from conservation of mass)

Eliminate y from the system

$$x + y = A \ \Rightarrow\ y = A - x$$

$$\frac{dx}{dt} = -k_1 x(A - x) = k_1 x(x - A)$$

This is a separable DE:

$$\int_{x_0}^{x} \frac{1}{x(x-A)}\, dx = \int_0^t k_1\, dt$$ **[2 pts]**

Integration by partial fractions:

$$\int_{x_0}^{x} \left(\frac{-1/A}{x} + \frac{1/A}{x-A}\right) dx = k_1 t$$

$$\frac{1}{A} \ln \frac{x-A}{x_0-A}\frac{x_0}{x} = k_1 t$$

$$\frac{x-A}{x_0-A}\frac{x_0}{x} = e^{Ak_1 t} \ \Rightarrow\ \frac{x_0}{x_0 - A}\left(1 - \frac{A}{x}\right) = e^{Ak_1 t}$$

General solution:

$$x = \frac{Ax_0}{x_0 - (x_0 - A)e^{Ak_1 t}}$$ **[2 pts]**

Particular solution:

$$A = x_0 + y_0 \ \Rightarrow\ x(t) = \frac{x_0(x_0 + y_0)}{x_0 + y_0\, e^{(x_0 + y_0)k_1 t}} = \frac{x_0 + y_0}{1 + \frac{y_0}{x_0}\, e^{(x_0 + y_0)k_1 t}}.$$

[1 pts]

$$y(t) = A - x(t) = x_0 + y_0 - x(t)$$

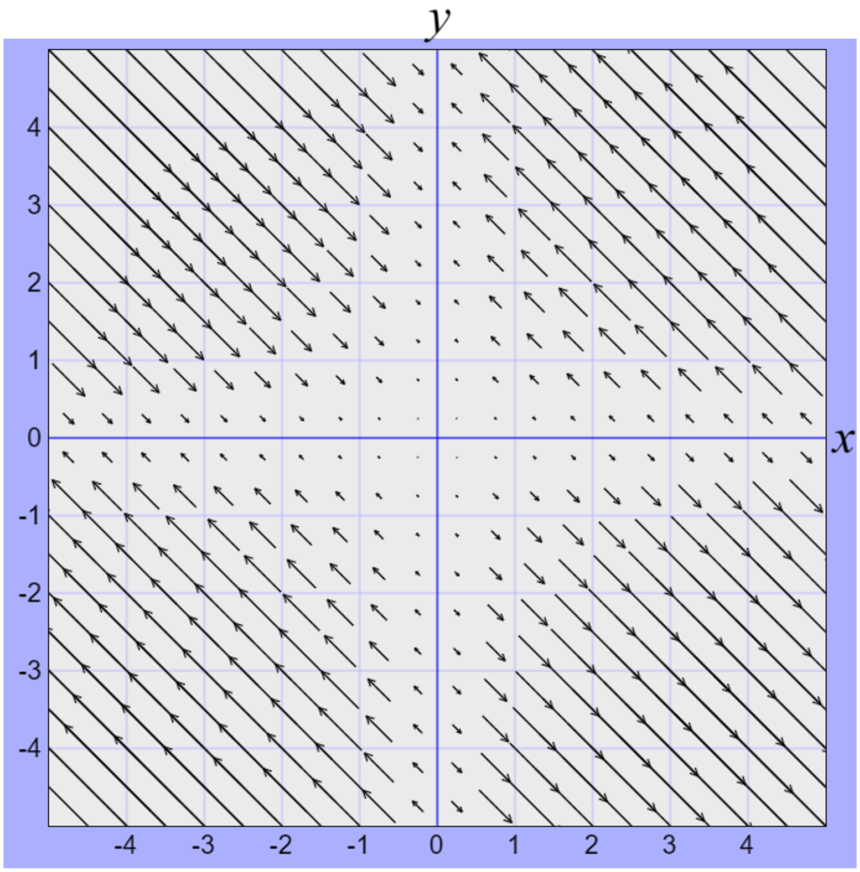
$$y(t) = (x_0 + y_0)\left(1 - \frac{1}{1 + \frac{y_0}{x_0}\, e^{(x_0 + y_0)k_1 t}}\right)$$

$$y(t) = \frac{(x_0 + y_0)\, y_0\, e^{(x_0 + y_0)k_1 t}}{x_0 + y_0\, e^{(x_0 + y_0)k_1 t}} = \frac{x_0 + y_0}{1 + \frac{x_0}{y_0}\, e^{-(x_0 + y_0)k_1 t}}.$$ **[2 pts]**

(Total: 7 points)

C4. (continued)

c. Phase plane for $x' = -k_1xy, \quad y' = k_1xy$:



[1 pt]

Nullclines: x-nullcline: $(x = 0 \;\; or \;\; y = 0)$
y-nullcline: $(x = 0 \;\; or \;\; y = 0)$ **[1 pt]**

Equilibrium points: $(x = 0 \; or \; y = 0)$ **[1 pt]**
(any point on the coordinate axes is an equilibrium point.)

The equilibrium points with $(x < 0, \;\; y = 0)$ and $(x = 0, \;\; y > 0)$ are stable.
The equilibrium points with $(x > 0, \;\; y = 0)$ and $(x = 0, \;\; y < 0)$ are unstable. **[1 pt]**

Only the region $x, y \geq 0$ is physically meaningful.

(Total: 4 points)

C4. (continued)

d. i) $x' = -k_1xy + k_2y^2, \ \ y' = k_1xy - k_2y^2 \ \Rightarrow \ x' + y' = 0$

$\Rightarrow \ x + y = A$, for some constant A. This is the same as the case $k_2 = 0$.

The system trajectories satisfy $x + y = A$ at all times **[1 pt]**, due to **conservation of mass [1 pt].**

(Total: 2 points)

ii) For the system $\frac{dx}{dt} = -k_1xy + k_2y^2, \ \frac{dy}{dt} = k_1xy - k_2y^2,$

x-nullcline: $y(k_2y - k_1x) = 0 \ \Rightarrow \ (y = 0 \ or \ y = \frac{k_1}{k_2}x)$

y-nullcline: $y(k_1x - k_2y) = 0 \ \Rightarrow \ (y = 0 \ or \ y = \frac{k_1}{k_2}x)$

The equilibrium regions are the lines $y = 0$ and $y = \frac{k_1}{k_2}x \ \Rightarrow \ k_2y = k_1x.$ **[1 pt]**

To prove stability, we can linearise the system and apply eigenvalue theory.

Let $\mathbf{x}_1 = \left(x_1, \ \frac{k_1}{k_2}x_1\right)$ be an arbitrary point on the equilibrium line, where $x_1 > 0$.

Let $f(x, y) = -k_1xy + k_2y^2 \ \Rightarrow \ \frac{\partial f}{\partial x} = -k_1y, \ \ \frac{\partial f}{\partial y} = -k_1x + 2k_2y.$

Let $g(x, y) = k_1xy - k_2y^2 \ \Rightarrow \ \frac{\partial g}{\partial x} = k_1y, \ \ \frac{\partial g}{\partial y} = k_1x - 2k_2y.$

Therefore, the Jacobian matrix of the system is

$$\mathbf{J_f}(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -k_1y & -k_1x + 2k_2y \\ k_1y & k_1x - 2k_2y \end{bmatrix}.$$ **[2 pts]**

Let $x = x_1$ and $y = \frac{k_1}{k_2}x_1$ to get the linearised system matrix $\mathbf{A}$:

$$\mathbf{A} = \mathbf{J_f}(x_1, \frac{k_1}{k_2}x_1) \begin{bmatrix} -\frac{k_1^2}{k_2}x_1 & k_1x_1 \\ \frac{k_1^2}{k_2}x_1 & -k_1x_1 \end{bmatrix} = k_1x_1 \begin{bmatrix} -k_1/k_2 & 1 \\ k_1/k_2 & -1 \end{bmatrix}.$$ **[1 pt]**

$\text{tr}\,\mathbf{A} = -k_1x_1\left(\frac{k_1}{k_2} + 1\right)$ and $\det \mathbf{A} = 0 \rightarrow$ eigenvalues $\lambda_1 = 0, \ \lambda_2 = -k_1x_1\left(\frac{k_1}{k_2} + 1\right).$

[1 pt]

Since all eigenvalues have $Re(\lambda) \leq 0$, these equilibria are **stable***.

Since one of the eigenvalues is zero, they are **degenerate fixed points. [1 pt]**

***Technicality:** *the 'Hartman-Grobman linearisation theorem', which allows us to use eigenvalue theory to infer the stability of nonlinear systems, does not strictly hold in our case since $Re(\lambda) = 0$. However, we have chosen to neglect this detail because we can already see that we have stability by inspecting the phase plane. 'Centre manifold theory' can be used for a formal proof.* **(Total: 6 points)**

C5.

a. Let $u = \frac{1}{r} \Rightarrow \frac{d^2u}{d\theta^2} + u = k$ where $k = \frac{gR^2}{r_0^{\,2} v_0^{\,2}}$.

General solution: $u(\theta) = A\cos\theta + B\sin\theta + k$ **[4 pts]**

Initial conditions: $u(0) = \frac{1}{r(0)} = \frac{1}{r_0} \Rightarrow \frac{1}{r_0} = A + k \Rightarrow A = \frac{1}{r_0} - k$

$u'(0) = \frac{-r'(0)}{r(0)} = 0 \Rightarrow B = 0$

Particular solution: $u(\theta) = \left(\frac{1}{r_0} - k\right)\cos\theta + k = k(1-\cos\theta) + \frac{1}{r_0}\cos\theta$ **[2 pts]**

Unsubstitute: $r(\theta) = \frac{1}{u(\theta)} = \frac{1}{k(1-\cos\theta) + \frac{1}{r_0}\cos\theta} = \frac{r_0^{\,2} v_0^{\,2}}{gR^2(1-\cos\theta) + r_0 v_0^{\,2}\cos\theta}$. **[1 pt]**

(Total: 7 points)

b. $\frac{du}{d\theta} = \left(k - \frac{1}{r_0}\right)\sin\theta = 0$ at maximum and minimum values of $r \Rightarrow \theta = 0$ or $\theta = \pi$

When $\theta = 0$, $r(0) = r_0$ (minimum) **[1 pt]**

When $\theta = \pi$, $r(\pi) = \frac{r_0}{2kr_0 - 1} = \frac{r_0^{\,2} v_0^{\,2}}{2gR^2 - r_0 v_0^{\,2}}$ (maximum) **[2 pts]**

(Total: 3 points)

c. i) Writing the 2nd order DE as a system of 1st order DEs,

$\frac{d^2u}{d\theta^2} + u = k$, let $v = \frac{du}{d\theta} \Rightarrow \left\{\frac{du}{d\theta} = v,\ \frac{dv}{d\theta} = k - u,\ u(0) = \frac{1}{r_0},\ v(0) = 0\right\}$

From Euler's method, we have $u_{n+1} = u_n + hv_n$ and $v_{n+1} = v_n + h\left(k - u_n\right)$. **[1 pt]**

This is a system of difference equations. Take the Z-transform of both equations:

$$z\,U(z) - \frac{z}{r_0} = U(z) + h\,V(z) \text{ and } z\,V(z) = V(z) - h\,U(z) + \frac{hk}{1-z^{-1}}$$ **[1 pt]**

Rearrange the second equation for $V(z)$: $V(z) = \frac{hk\,z}{(z-1)^2} - \frac{h\,U(z)}{z-1}$

Substitute into the first equation to eliminate $V(z)$: $z\,U(z) - \frac{z}{r_0} = U(z) + \frac{h^2 k\,z}{(z-1)^2} - \frac{h^2\,U(z)}{z-1}$

Solve for $U(z)$ and factorise: $U(z) = \frac{z\left((z-1)^2 + h^2 k r_0\right)}{r_0(z-1)\left((z-1)^2 + h^2\right)}$ **[3 pts]**

The poles of $U(z)$ are at $z = 1$ and $z = 1 \pm ih$.

The zeroes of $U(z)$ are at $z = 0$ and $z = 1 \pm ih\sqrt{kr_0}$. **[1 pt]**

If $gR^2 = r_0 v_0^{\,2} \Rightarrow kr_0 = 1 \Rightarrow$ pole and zero cancel $\rightarrow U(z) = \frac{z}{r_0(z-1)}$. **[1 pt]**

This corresponds to a circular orbit around the Earth, where $u_n = 1/r_0$ is a constant.
The approximation for u_n is exact in this case ($u_n = u(nh)$). **(Total: 9 points)**

ii) Initial value theorem: $\lim_{z\to\infty} \frac{z(z-1)^2 + h^2 k r_0 z}{r_0(z-1)\left((z-1)^2 + h^2\right)} = \lim_{z\to\infty} \frac{z^3\left(1 + O(z^{-1})\right)}{r_0 z^3\left(1 + O(z^{-1})\right)} = \frac{1}{r_0} = u_0$. **[1 pt]**

(where $O(\cdot)$ is the 'Big O asymptotic notation'.) **(Total: 1 point)**

C6.

a. Resolving forces on mass m_1:

$$m_1 y_1'' = k_2(y_2 - y_1) + \lambda(y_2' - y_1') - k_1 y_1 - \lambda y_1' + f_1(t)$$

Resolving forces on mass m_2:

$$m_2 y_2'' = -k_1 y_2 + k_2(y_1 - y_2) + \lambda(y_1' - y_2')$$

Expand and convert to standard form:

$$m_1 y_1'' + 2\lambda y_1' - \lambda y_2' + (k_1 + k_2)y_1 - k_2 y_2 = f_1(t) \quad \textbf{[3 pts]}$$

$$m_2 y_2'' - \lambda y_1' + \lambda y_2' - k_2 y_1 + (k_1 + k_2)y_2 = 0 \quad \textbf{[2 pts]}$$

Writing in matrix form: **[3 pts]**

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} + \underbrace{\begin{bmatrix} 2\lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}$$

(Total: 8 points)

b. Let $v_1 = y_1'$ and $v_2 = y_2'$. **[1 pts]**

$\Rightarrow \quad m_1 v_1' = -2\lambda v_1 + \lambda v_2 - (k_1 + k_2)y_1 + k_2 y_2 + f_1(t)$

and $m_2 v_2' = \lambda v_1 - \lambda v_2 + k_2 y_1 - (k_1 + k_2)y_2$.

Therefore,

$$\begin{bmatrix} y_1' \\ y_2' \\ v_1' \\ v_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{2\lambda}{m_1} & \frac{\lambda}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{\lambda}{m_2} & -\frac{\lambda}{m_2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ v_1 \\ v_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{f_1(t)}{m_1} \\ 0 \end{bmatrix}}_{\mathbf{f}} \quad \textbf{[5 pts]}$$

(Total: 6 points)

C6. (continued)

c. i.

For a system of DEs, the complementary solution is given by

$$\mathbf{x}(t) = \begin{cases} c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2 & \text{if } \lambda_{1,2} \text{ are real} \\ c_1 e^{\alpha t}(\mathbf{u}_1 \cos\beta t + \mathbf{u}_2 \sin\beta t) + c_2 e^{\alpha t}(\mathbf{u}_1 \cos\beta t - \mathbf{u}_2 \sin\beta t) & \text{if } \lambda_{1,2} = \alpha \pm \beta i \text{ are complex} \\ c_1 e^{\lambda t}\mathbf{u} + c_2 e^{\lambda t}(\mathbf{u}t + \mathbf{v}), \text{ for any } \mathbf{v} : (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{u} & \text{if } \lambda \text{ is a repeated defective eigenvalue} \end{cases}$$

Given that there are two pairs of unequal complex conjugate eigenvalues, we have

$$\mathbf{x}_{CF}(t) = c_1 e^{\alpha t}(\mathbf{u}_1 \cos\beta t + \mathbf{u}_2 \sin\beta t) + c_2 e^{\alpha t}(\mathbf{u}_1 \cos\beta t - \mathbf{u}_2 \sin\beta t) \\ + c_3 e^{\gamma t}(\mathbf{u}_3 \cos\delta t + \mathbf{u}_4 \sin\delta t) + c_4 e^{\gamma t}(\mathbf{u}_3 \cos\delta t - \mathbf{u}_4 \sin\delta t)$$

where c_1, c_2, c_3, c_4 are arbitrary real constants. **[4 pts]**

(Total: 4 points)

ii.

Let

$$\mathbf{x}_1 = e^{\alpha t}(\mathbf{u}_1 \cos\beta t + \mathbf{u}_2 \sin\beta t)$$
$$\mathbf{x}_2 = e^{\alpha t}(\mathbf{u}_1 \cos\beta t - \mathbf{u}_2 \sin\beta t)$$
$$\mathbf{x}_3 = e^{\gamma t}(\mathbf{u}_3 \cos\delta t + \mathbf{u}_4 \sin\delta t)$$
$$\mathbf{x}_4 = e^{\gamma t}(\mathbf{u}_3 \cos\delta t - \mathbf{u}_4 \sin\delta t)$$

These are the four linearly independent vector-valued basis functions for $\mathbf{x}_{CF}(t)$.

Let $\mathbf{X}$ be the 4 × 4 matrix with columns $[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \mathbf{x}_4]$. **[1 pt]**

Then, by variation of parameters for systems,

$$\mathbf{x}_{PI}(t) = \mathbf{X} \int \mathbf{X}^{-1}\ \mathbf{f}(t)\ dt$$ **[1 pt]**

(Total: 2 points)

C7.

a. Let $f(x, y) = \sqrt{|x|}$. This function is continuous for all real x.

We need to prove that there exists some x_1, x_2 such that $\frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$ diverges.

Observe that if we let $x_2 = x_1 + h$, then the expression is $\left|\frac{f(x_1 + h) - f(x_1)}{h}\right|$.

If we let $x_1 = 0$ and take the limit as $h \to 0$, we get

$$\lim_{h \to 0} \left|\frac{f(x_1 + h) - f(x_1)}{h}\right| = \lim_{h \to 0} \left|\frac{\sqrt{|h|}}{h}\right| = \lim_{h \to 0} \frac{1}{\sqrt{|h|}} = \infty$$ (diverges). **[3 pts]**

(Alternatively, this expression is the definition of the derivative $f'(0)$, which is undefined.)

Therefore, it is impossible to find a finite K such that $\frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} \leq K$ in this case.

Therefore, the function $f(x) = \sqrt{|x|}$ is **not** Lipschitz continuous. **[1 pt]**

(Total: 4 points)

b. Starting with $r'' = \sqrt{|r|}$, $r(0) = r'(0) = 0$, substitute $v = r'$ to convert to a system of two 1st-order DEs:

$$\Rightarrow \left\{v' = \sqrt{|r|},\ \ r' = v,\ \ v(0) = 0,\ \ r(0) = 0\right\}.$$ **[1 pt]**

Let the vector $\mathbf{x} = [v, r]^{\mathsf{T}}$. Then the system is $\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)$, where $\mathbf{f}(v, r, t) = \begin{bmatrix} \sqrt{|r|} \\ v \end{bmatrix}$ is a multivariable vector-valued function. **[1 pt]**

$\mathbf{f}(v, r, t)$ is continuous in t, since it is independent of t.
To prove that $\mathbf{f}$ is not Lipschitz continuous in $\mathbf{x}$, we need to prove that there exists some v_1, v_2, r_1, r_2 such that $\left|\frac{\mathbf{f}(v_1, r_1) - \mathbf{f}(v_2, r_2)}{\sqrt{(v_1 - v_2)^2 + (r_1 - r_2)^2}}\right|$ diverges.

Choose $v_1 = v_2 = 0$, let $r_1 = 0$ and take $r_2 = r_1 + h$ in the limit as $h \to 0$. This leads to the same expression as in part **a)**, so this function is **not** Lipschitz continuous. **[2 pts]**

Therefore, the Picard-Lindelöf theorem is **not** satisfied (no unique solution). **[1 pt]**

(Total: 5 points)

c.

i) Differentiate the given candidate solution:

$$r(t) = \left\{0,\ t \le T; \quad \frac{1}{144}(t - T)^4,\ \ t > T\right\}$$
$$\Rightarrow\ r'(t) = \left\{0,\ t \le T; \quad \frac{1}{36}(t - T)^3,\ \ t > T\right\}$$
$$\Rightarrow\ r''(t) = \left\{0,\ t \le T; \quad \frac{1}{12}(t - T)^2,\ \ t > T\right\}$$ **[1 pt]**

For $t \le T$, the differential equation is

$r'' = \sqrt{|r|}\ \Rightarrow\ 0 = \sqrt{|0|} = 0$ (correct: DE is satisfied) **[1 pt]**

For $t > T$, the differential equation is

$r'' = \sqrt{|r|}\ \Rightarrow\ \frac{1}{12}(t - T)^2 = \sqrt{\left|\frac{1}{144}(t - T)^4\right|} = \frac{1}{12}(t - T)^2$ (correct: DE is satisfied)

Therefore, for all t, LHS = RHS, so the given solution satisfies the DE for all t. **[1 pt]**

(Total: 3 points)

ii) ‘**Causal**’ means that a system’s behaviour at time $t = a$ is not dependent on its behaviour at any times $t > a$.

However, for Norton’s dome, **if** we suppose that

$$r(t) = \begin{cases} 0 & t \leq T, \\ \frac{1}{144}(t-T)^4 & t > T \end{cases}$$

is a physically valid solution for the particle under Newtonian mechanics, then the behaviour at time $t = a$ is dependent on whether $a < T$ or $a > T$, and since T is unknown, the system is **non-deterministic**, and appears to be **non-causal.**

However, it is fallacious to claim that Newtonian mechanics *does* permit this solution. The violation of the Picard-Lindelöf theorem in part **b)** is what gives multiple possible solutions to this DE, but this is a purely mathematical model and it is **not** the case that physical reality (nor reality as modelled by Newtonian mechanics) can follow any of these solutions.

It can only be said that **Newton’s second law** (used to derive the DE) is *not fully descriptive* of physical reality. This statement is obvious when we consider that reality is *also* bound by Newton’s first law as well. When we apply **Newton’s first law**, we see that $r(t) = 0$ is the **only** solution satisfying both constraints for all time t.

Therefore, we may say that under Newton’s second law *only*, the system is non-causal, but Newtonian mechanics as a whole is causal and deterministic (at least, in this case!).

There is room for subjectivity and opinion in this answer - additionally, this answer in itself can be scrutinised: Norton’s dome has caused considerable debate! You may find the following discussions interesting:

- *John Norton’s claim about Norton’s dome: here*
- *A refutation by Gruff Davies: here*

Mark this question according to how many of the various talking points you considered in these articles.

(Total: 8 points)

C8.

a. In standard form, Bessel's equation for the zeroth order is $y'' + \frac{1}{x}y' + y = 0$.
There is one singular point at $x = 0$.
Since $u(x) = x \times \frac{1}{x} = 1$ and $v(x) = x^2 \times 1 = x^2$ are both smooth functions, the point at $x = 0$ is a regular singular point. Therefore, the Frobenius method can be used about $x = 0$.

Taylor series: $u(x) = 1$ and $v(x) = 0 + x^2 \Rightarrow u_0 = 1,\ v_0 = 0.$

Indicial equation: $r(r-1) + u_0 r + v_0 = 0 \Rightarrow r^2 = 0$
$\Rightarrow r = 0$ (repeated root).

General solution: $y = (A + B\ln x)\sum_{k=0}^{\infty} a_k x^k + B\sum_{k=1}^{\infty} b_k x^k$ for $x > 0$.

First basis function: $y_1 = \sum_{k=0}^{\infty} a_k x^k$ **[1 pt]**

To find a_k, since the root is $r = 0$, the first basis solution is equivalent to a Maclaurin series, so we can use the Leibniz-Maclaurin method. Differentiate both sides of the given DE with respect to x, k times using the general Leibniz rule for product terms:

$$\frac{d^k}{dx^k}\left(x^2 y^{(2)} + xy^{(1)} + x^2 y\right) = \left(x^2 y^{(k+2)} + 2kxy^{(k+1)} + k(k-1)y^{(k)}\right)$$
$$+ \left(xy^{(k+1)} + ky^{(k)}\right) + \left(x^2 y^{(k)} + 2kxy^{(k-1)} + k(k-1)y^{(k-2)}\right)$$
$$= x^2 y^{(k+2)} + (2k+1)xy^{(k+1)} + (k^2 + x^2)y^{(k)}$$
$$+ 2kxy^{(k-1)} + k(k-1)y^{(k-2)} = 0$$

Let $x = 0$: $k^2 y^{(k)}(0) + k(k-1)\, y^{(k-2)}(0) = 0$

Coefficients: $k^2\, k!\, a_k + k(k-1)(k-2)!\, a_{k-2} = 0$
$\Rightarrow k^2\, k!\, a_k + k!\, a_{k-2} = 0 \Rightarrow a_k = \frac{-1}{k^2} a_{k-2}$. **[4 pts]**

Initial conditions: $a_0 = 1,\ a_1 = 0 \Rightarrow$ all odd terms are zero. **[1 pt]**

Therefore, $a_0 = 1,\ a_2 = \frac{-1}{2^2},\ a_4 = \frac{-1}{4^2} \times \frac{-1}{2^2},\ a_6 = \frac{-1}{6^2} \times \frac{-1}{4^2} \times \frac{-1}{2^2}$...

The pattern is $a_{2k} = \frac{(-1)^k}{(2k!!)^2} = \frac{(-1)^k}{(2^k k!)^2} = \frac{(-1)^k}{4^k (k!)^2} \Rightarrow J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2} x^{2k}$. **[1 pt]**

(Total: 7 points)

C8. (continued)

b. First basis solution: $y_1 = \sum_{k=0}^{\infty} a_k x^{k+r}$

Differentiate: $\Rightarrow y_1' = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}$

$\Rightarrow y_1'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}$

Sub into DE:

$$\Rightarrow \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + \frac{1}{x}\sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} + \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Absorb powers of x:

$$\Rightarrow \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-2} + \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Re-index to make exponents of x all the same:

$$\Rightarrow \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} + \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-2} + \sum_{k=2}^{\infty} a_{k-2} x^{k+r-2} = 0$$

Pull out first two terms to make index k start at the same value in all summations:

$$\Rightarrow r(r-1)a_0 x^{r-2} + (r+1)r a_1 x^{r-1} + r a_0 x^{r-2} + (r+1) a_1 x^{r-1} + \sum_{k=2}^{\infty} \dots + \sum_{k=2}^{\infty} \dots + \sum_{k=2}^{\infty} \dots = 0$$

The indicial equation for the x^{r-1} coefficient is $\left(r^2 + 2r + 1\right)a_1 = 0$,

so $a_1(r) = 0$ for all values of r (except $r = -1$, where a_1 is free).

Combine summations:

$$\Rightarrow \dots + \sum_{k=2}^{\infty} \left((k+r)(k+r-1) a_k + (k+r) a_k + a_{k-2}\right) x^{k+r-2} = 0$$

General recurrence relation: $(k+r)(k+r-1) a_k + (k+r) a_k + a_{k-2} = 0$

$\Rightarrow (k+r)^2 a_k + a_{k-2} = 0 \Rightarrow a_k = \frac{-1}{(k+r)^2} a_{k-2}$ **[8 pts]**

Therefore, for even $k = 2m$, $a_{2m} = \frac{-1}{(2m+r)^2} \frac{-1}{(2(m-1)+r)^2} \dots \frac{-1}{(2+r)^2} a_0$.

From part **a)**, we found $a_0 = 1$: $a_{2m} = \frac{1}{(2m+r)^2} \frac{1}{(2(m-1)+r)^2} \dots \frac{1}{(2+r)^2} (-1)^m$. **[1 pt]**

C8. (continued)

To find the coefficients of the second basis solutions, we use $b_k = \frac{da_k}{dr}(0)$.

From the indicial equation, we have $b_1 = \frac{d}{dr}(a_1(0)) = 0$.

For odd k, we have $a_k = 0$, so $b_k = 0$.

For even $k = 2m$, we have $b_{2m} = \frac{d}{dr}\left(\frac{1}{(2m+r)^2}\ \frac{1}{(2(m-1)+r)^2} \cdots \frac{1}{(2+r)^2}\right)(-1)^m a_0$ at $r = 0$.

We need to differentiate this product. However, it will be easier to use logarithmic differentiation. Consider $\frac{d}{dr}\ln\left|a_{2m}(r)\right| = \frac{a_{2m}'(r)}{a_{2m}(r)} = \frac{b_{2m}(r)}{a_{2m}(r)}$.

By log identities, we have

$\ln\left|a_{2m}(r)\right| = -2\left(\ln|2m+r| + \ln|2(m-1)+r| + \ldots + \ln|2+r|\right)$

Differentiating, $\frac{d}{dr}\ln\left|a_{2m}(r)\right| = -2\left(\frac{1}{2m+r} + \frac{1}{2(m-1)+r} + \ldots + \frac{1}{2+r}\right)$

At $r = 0$, we get $\frac{d}{dr}\ln\left|a_{2m}(0)\right| = -\sum_{i=1}^{m}\frac{1}{i} = -H_m$ and $a_{2m}(0) = \frac{(-1)^m}{4^m (m!)^2}$.

Therefore, $b_{2m} = \frac{(-1)^m}{4^m (m!)^2} \times -H_m = \frac{(-1)^{m+1}}{4^m (m!)^2} H_m$. **[4 pts]**

Our coefficients are therefore $b_k = \begin{cases} 0, & k \text{ odd} \\ \frac{(-1)^{m+1}}{4^m (m!)^2} H_m, & k \text{ even},\ k = 2m \end{cases}$

(Total: 13 points)

C8. (continued)

c.

i) **No.** Despite both being linearly independent of $J_0(x)$, the basis solutions $y_2(x)$ and $Y_0(x)$ are **not necessarily equal** because the boundary conditions for $Y_0(x)$ are not specified, so the undetermined coefficients for $Y_0(x)$ may be different from those of $y_2(x)$. **[1 pt]**

(*In reality,* $Y_0(x)$ *is defined as* $Y_0(x) \approx 0.63662\, y_2(x) - 0.07381\, J_0(x)$ *so* $Y_0(x) \neq y_2(x)$.)

(Total: 1 point)

ii) **No.** The Bessel DE for order 0 can be written $y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0.$

It can be seen that $x = 0$ is the only singular point, and $x = 0$ is a regular singular point. So by Fuchs' theorem, the radius of convergence for the series

$$\sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad \sum_{k=1}^{\infty} b_k x^k$$

are both infinity (i.e. all $x \in \mathbf{C}$). However, since y_2 includes a $\ln x$ term, which has a branch point at $x = 0$, this solution is **only defined for real** $x > 0$. **[1 pt]**

(*If the complex definition of the logarithm was used:* $\ln z = \ln r + i(\theta + 2\pi n)$*, then the solution for* $y_2(z)$ *would be defined for all complex* $z \neq 0$.)

(Total: 1 point)

C8. (continued)

d.

i) Since $y = J_n(x)$ satisfies the first differential equation, let $z = x^n y$.

Differentiate: $z = x^n y \ \Rightarrow\ z' = nx^{n-1} y + x^n y'$

$\Rightarrow\ z'' = n(n-1)\,x^{n-2}y + 2nx^{n-1}y' + x^n y''$ **[1 pt]**

Substitute into the given DE:

$$\Rightarrow\ x\left(n(n-1)\,x^{n-2}y + 2nx^{n-1}y' + x^n y''\right) + (1-2n)\left(nx^{n-1}y + x^n y'\right) + x^{n+1}y = 0$$
$$\Rightarrow\ n(n-1)\,x^{n-1}y + 2nx^n y' + x^{n+1}y'' + n(1-2n)x^{n-1}y + (1-2n)x^n y' + x^{n+1}y = 0$$
$$\Rightarrow\ n(n-1)y + 2nxy' + x^2 y'' + n(1-2n)y + (1-2n)xy' + x^2 y = 0$$
$$\Rightarrow\ x^2 y'' + xy' + \left(x^2 - n^2\right)y = 0$$ **[3 pts]**

This is the original Bessel differential equation, which $y = J_n(x)$ is defined to satisfy, so the original substitution also satisfied the given differential equation.

(Total: 4 points)

ii) Let $n = \frac{1}{2}$ in the given DE. Then, for suitable boundary conditions, $z(x) = \sqrt{x}\,J_{1/2}(x)$ is the particular solution to the differential equation $x\,z'' + x\,z = 0$.

If we remove $x = 0$ from the domain of $z(x)$, we can simplify this to $z'' + z = 0$.
This is the simple harmonic motion DE. The general solution is $z(x) = A\cos x + B\sin x$.

Therefore, $J_{1/2}(x) = A\frac{\cos x}{\sqrt{x}} + B\frac{\sin x}{\sqrt{x}}$ for some real constants A and B. **[2 pts]**

Since $n = \frac{1}{2}$ is not an integer, we **cannot** use the given initial conditions for $J_n(x)$.
However, we are given that $\lim_{x\to 0^+} J_{1/2}(x)$ is finite.

Therefore, $\lim_{x\to 0^+}\left(A\frac{\cos x}{\sqrt{x}} + B\frac{\sin x}{\sqrt{x}}\right) = A\lim_{x\to 0^+}\frac{\cos x}{\sqrt{x}} + B\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}$.

Since $\lim_{x\to 0^+}\frac{\cos x}{\sqrt{x}}$ does not exist (diverges to ∞), we must have $A = 0$. **[1 pt]**

Since $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+}\frac{x - O(x^3)}{x^{1/2}} = \lim_{x\to 0^+} x^{1/2} + O(x^{5/2}) = 0$, B may be any constant.

Therefore, $J_{1/2}(x) = B\frac{\sin x}{\sqrt{x}}$, which is proportional to $\frac{\sin x}{\sqrt{x}}$. **[1 pt]**

(Total: 4 points)

C9

a. In spherical coordinates, we have $d\mathbf{r} = dr\ \hat{\mathbf{r}} + r\sin\phi\ d\theta\ \hat{\boldsymbol{\theta}} + rd\phi\ \hat{\boldsymbol{\phi}}$.

Therefore, the arc length differential element is

$$dS^2 = |d\mathbf{r}|^2 = dr^2 + (r\ sin\ \phi)^2\ d\theta^2 + (r\ d\phi)^2$$

The total arc length is then $S = \int_0^{2\pi} \frac{dS}{d\theta}\ d\theta = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2\ sin^2\ \phi + r^2\left(\frac{d\phi}{d\theta}\right)^2}\ d\theta$ **[2 pts]**

On the cone's surface, ϕ is constant so $\frac{d\phi}{d\theta} = 0 \Rightarrow S = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2\ sin^2\ \phi}\ d\theta$

The value of ϕ is the half-angle of the cone, which is $\phi = sin^{-1}\frac{1}{3}$, so

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{9}r^2}\ d\theta$$

Substitute $u = \frac{dr}{d\theta} \Rightarrow S = \int_0^{2\pi} \sqrt{u^2 + \frac{1}{9}r^2}\ d\theta = \int_0^{2\pi} \frac{1}{3}\sqrt{9u^2 + r^2}\ d\theta$.

Therefore $S = \int_0^{2\pi} g(r,\ u)\ d\theta$ where $g(r,\ u) = \frac{1}{3}\sqrt{9u^2 + r^2}$. **[2 pts]**

(Total: 4 points)

C9. (continued)

b. Starting with $g(r,\, u) = \frac{1}{3}\sqrt{9u^2 + r^2}$,

Evaluate the partial derivatives of g: $\frac{\partial g}{\partial r} = \frac{1}{3} \times \frac{r}{\sqrt{9u^2+r^2}}$ and $\frac{\partial g}{\partial u} = \frac{1}{3} \times \frac{9u}{\sqrt{9u^2+r^2}}$ **[2 pts]**

Substitute into the Euler-Lagrange differential equation:

$$\frac{1}{3} \times \frac{r}{\sqrt{9u^2+r^2}} = \frac{d}{d\theta}\left(\frac{1}{3} \times \frac{9u}{\sqrt{9u^2+r^2}}\right) \Rightarrow \frac{r}{\sqrt{9u^2+r^2}} = \frac{d}{d\theta}\left(\frac{9u}{\sqrt{9u^2+r^2}}\right).$$

Using the quotient rule, $\frac{d}{d\theta}\left(\frac{9u}{\sqrt{9u^2+r^2}}\right) = \frac{9\frac{du}{d\theta}\sqrt{9u^2+r^2} - 9u\frac{18u\frac{du}{d\theta} + 2r\frac{dr}{d\theta}}{2\sqrt{9u^2+r^2}}}{9u^2+r^2}$, so the equation is

$$\Rightarrow \frac{r}{\sqrt{9u^2+r^2}} = \frac{9\frac{du}{d\theta}\sqrt{9u^2+r^2} - 9u\frac{18u\frac{du}{d\theta} + 2r\frac{dr}{d\theta}}{2\sqrt{9u^2+r^2}}}{9u^2+r^2}$$

Multiply both sides by $\left(9u^2 + r^2\right)^{3/2}$:

$\Rightarrow r\left(9u^2 + r^2\right) = 9\frac{du}{d\theta}\left(9u^2 + r^2\right) - \frac{9}{2}u\left(18u\frac{du}{d\theta} + 2r\frac{dr}{d\theta}\right)$ **[3 pts]**

Unsubstitute $u = \frac{dr}{d\theta} \Rightarrow \frac{du}{d\theta} = \frac{d^2r}{d\theta^2}$, and denote these as r' and r'':

$\Rightarrow 9r(r')^2 + r^3 = 81(r')^2r'' + 9r^2r'' - 81(r')^2r'' - 9r(r')^2$

$\Rightarrow r^3 + 18(r')^2r = 9r^2r''$

$\Rightarrow r'' - \frac{2}{r}(r')^2 - \frac{1}{9}r = 0$ (for $r \neq 0$) **[2 pts]**

This is a nonlinear second-order DE. The nonlinearity $\frac{1}{r}(r')^2$ suggests a substitution:

Let $v = (r')^2$, where v is a function of θ. Differentiating both sides w.r.t. θ,

$$\frac{dv}{d\theta} = 2r'r'' \text{ and } \frac{dv}{dr} = \frac{v'}{r'} = 2r'' \Rightarrow r'' = \frac{1}{2}\frac{dv}{dr}.$$

Applying these substitutions, the DE becomes $\frac{1}{2}\frac{dv}{dr} - \frac{2}{r}v - \frac{1}{9}r = 0$.

Therefore, $\frac{dv}{dr} - \frac{4}{r}v = \frac{2}{9}r$. This is a first-order linear DE. **[2 pts]**

Integrating factor method: $I(r) = r^{-4} \Rightarrow r^{-4}v = \int \frac{2}{9}r^{-3}\,dr \Rightarrow v = \frac{-r^2}{9} + Cr^4$

Unsubstitute: $\frac{dr}{d\theta} = \sqrt{v} = \sqrt{Cr^4 - \frac{1}{9}r^2} = \frac{1}{3}r\sqrt{9Cr^2 - 1}$

C9. (continued)

b. This is a separable DE: $\int \frac{1}{r\sqrt{9Cr^2-1}}\,dr = \int \frac{1}{3}\,d\theta$ **[2 pts]**

To evaluate the integral $\int \frac{1}{r\sqrt{9Cr^2-1}}\,dr = \int \frac{1}{3\sqrt{C}\,r\sqrt{r^2-\frac{1}{9C}}}$, let $a^2 = \frac{1}{9C}$ and use the integral table result, $\int \frac{a}{\sqrt{x^4-a^2x^2}}\,dx = sec^{-1}\,\frac{x}{a} \Rightarrow \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\,sec^{-1}\,\frac{x}{a}$ (for $x > 0$).

The integral is then $\int \frac{1}{3\sqrt{C}\,r\sqrt{r^2-\frac{1}{9C}}}\,dr = sec^{-1}(3\sqrt{C}r) + D$.

The differential equation becomes $sec^{-1}(3\sqrt{C}r) = \frac{1}{3}\theta + D$.

General solution: $r = A\,sec\left(\frac{1}{3}\theta + B\right)$ (rename: $A = \frac{1}{3\sqrt{C}}$ and $B = D$.) **[4 pts]**

Boundary conditions: $r(0) = 60,\ \ r(2\pi) = 50$.

Therefore $60 = A\,sec\,B$ and $50 = A\,sec\left(\frac{2\pi}{3} + B\right)$

$$\Rightarrow \frac{sec\left(\frac{2\pi}{3}+B\right)}{sec\,B} = \frac{5}{6} \Rightarrow \frac{cos\,B}{cos\left(\frac{2\pi}{3}+B\right)} = \frac{5}{6}$$

$$\Rightarrow 6\,cos\,B = 5\left(cos\,\frac{2\pi}{3}\,cos\,B - sin\,\frac{2\pi}{3}\,sin\,B\right)$$

$$\Rightarrow 6\,cos\,B = -\frac{5}{2}\,cos\,B - \frac{5\sqrt{3}}{2}\,sin\,B$$

$$\Rightarrow \frac{17}{2}\,cos\,B = -\frac{5\sqrt{3}}{2}\,sin\,B$$

$$\Rightarrow tan\,B = -\frac{17\sqrt{3}}{15} \Rightarrow B = -\,tan^{-1}\,\frac{17\sqrt{3}}{15}$$

$$\Rightarrow A = 60\,cos\left(tan^{-1}\,\frac{17\sqrt{3}}{15}\right) = 150\sqrt{\frac{3}{91}}.$$

Particular solution: $r(\theta) = 150\sqrt{\frac{3}{91}}\,sec\left(\frac{1}{3}\theta - tan^{-1}\,\frac{17\sqrt{3}}{15}\right)$. **[2 pts]**

To evaluate the overall arc length, use $r = A\,sec\left(\frac{1}{3}\theta + B\right)$

$$\Rightarrow r^2 = A^2\,sec^2\left(\frac{1}{3}\theta + B\right) \text{ and } u = r' = \frac{1}{3}A\,sec\left(\frac{1}{3}\theta + B\right)tan\left(\frac{1}{3}\theta + B\right)$$

$$\Rightarrow 9u^2 = A^2\,sec^2\left(\frac{1}{3}\theta + B\right)tan^2\left(\frac{1}{3}\theta + B\right)$$

Therefore

$$S = \int_0^{2\pi} \frac{1}{3}\sqrt{9u^2 + r^2}\,d\theta = \frac{1}{3}\int_0^{2\pi} \sqrt{A^2\,sec^2\left(\frac{1}{3}\theta + B\right)tan^2\left(\frac{1}{3}\theta + B\right) + A^2\,sec^2\left(\frac{1}{3}\theta + B\right)}\,d\theta$$

C9. (continued)

b. Factorise and use the Pythagorean trig identity:

$$S = \frac{A}{3}\int_0^{2\pi} sec^2\left(\frac{1}{3}\theta + B\right) d\theta = A\left(tan\left(\frac{2\pi}{3} + B\right) - tan\, B\right)$$

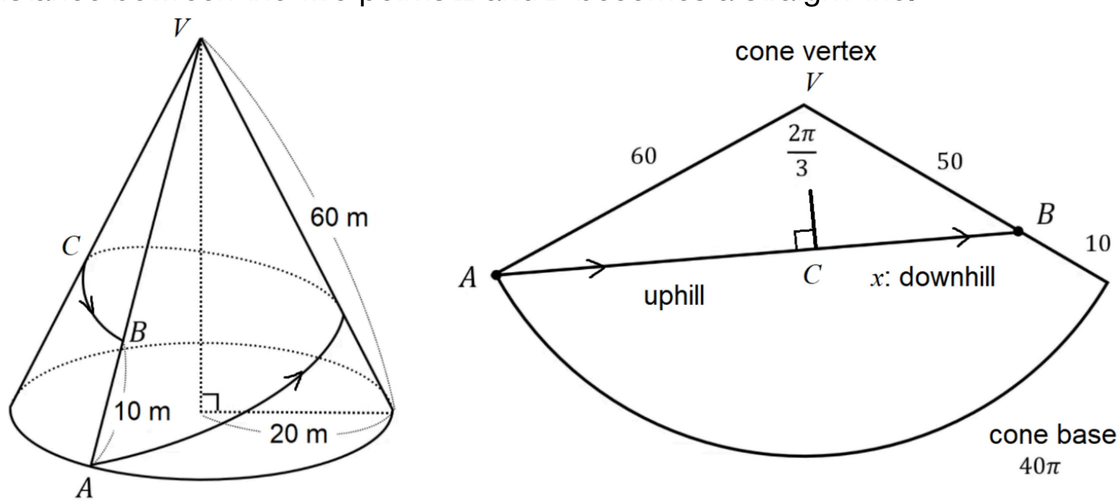
Use tangent addition identity: $S = A\left(\frac{-\sqrt{3} + tan\, B}{1 + \sqrt{3}\, tan\, B} - tan\, B\right)$

Replace the constants. Let $tan\, B = \frac{-17\sqrt{3}}{15}$ and $A = 150\sqrt{\frac{3}{91}}$:

$\Rightarrow\ S = 150\sqrt{\frac{3}{91}}\left(\frac{8\sqrt{3}}{9} + \frac{17\sqrt{3}}{15}\right) = 150\sqrt{\frac{3}{91}} \times \frac{91\sqrt{3}}{45} = 10\sqrt{91}$. **[3 pts]**

(Total: 20 points)

c. Consider the net of the cone, which unwraps into a circular sector. The shortest distance between the two points A and B becomes a straight line.



The arc length of the sector is the circumference of the base = $2\pi \times 20 = 40\pi$.
The angle of the sector satisfies $60 \times \theta = 40\pi \Rightarrow \theta = \frac{2\pi}{3}$ radians.

By cosine rule in $\triangle ABV$, $|AB| = \sqrt{50^2 + 60^2 - 2 \times 50 \times 60 \times cos\,\frac{2\pi}{3}} = 10\sqrt{91}$. **[3 pts]**
This matches the value of S found in part **b)** (the total length of the train track).

The downhill section begins at the point on the line for which the radius passing through the point is perpendicular to the line AB. Let this point be C, and the downhill distance $|BC| = x$, so $|AC| = 10\sqrt{91} - x$. Let the distance between C and the apex (sector centre) be h.

Pythagoras in $\triangle ACV$ and $\triangle BCV$: $(10\sqrt{91} - x)^2 + h^2 = 60^2$ and $x^2 + h^2 = 50^2$

$\Rightarrow (10\sqrt{91} - x)^2 - x^2 = 1100 \Rightarrow 20\sqrt{91}x = 8000 \Rightarrow x = \frac{400}{\sqrt{91}}$ metres. **[3 pts]**

(Total: 6 points)

Video Solutions for Some Questions

C2. Partial solution by blackpenredpen on YouTube:
solving an almost-exact differential equation (with a special integrating fact…

C9c. Geometric solution for geodesics on a cone by MindYourDecisions on YouTube:
VERY HARD South Korean Geometry Problem (CSAT Exam)